

# The binary black-hole dynamics at the third-and-a-half post-Newtonian order in the ADM-formalism

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We specialize the radiation-reaction part of the Arnowitt-Deser-Misner (ADM) Hamiltonian for many non-spinning point-like bodies, calculated by Jaranowski and Schäfer [1], to third-and-a-half post-Newtonian approximation to general relativity, to binary systems. This Hamiltonian is used for the computation of the instantaneous gravitational energy loss of a binary to 1PN reactive order. We also derive the equations of motion, which include PN reactive terms via Hamiltonian and Euler-Lagrangian approaches. The results are consistent with the expressions for reactive acceleration provided by Iyer-Will formalism in Ref. [2] in a general class of gauges.

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## I. INTRODUCTION

Compact binaries are the most promising sources of detectable gravitational waves for ground and space based interferometric experiments, TAMA300, GEO600, VIRGO, LIGO (Laser Interferometer Gravitational Wave Observatory) and LISA (Laser Interferometer Space Antenna). Because of gravitational wave emission, the binary separation decreases adiabatically in time while the frequency increases according to the relativistic version of the Kepler law. This process ends as soon as the first instable orbit is reached before the coalescence. During the inspiraling phase the binary system produces a most likely detectable signal. The construction of wave patterns for data analysis requires the knowledge of the theoretical wave form to very high post-Newtonian orders [see, e.g., Refs. [3, 4]]. This assumes a good understanding of the compact-binary dynamics. The secular effects governing the long-time evolution are of particular importance. They result from the reaction force due to the quadrupole damping arising at the 2.5 order in power of  $v^2/c^2$ ,  $v$  representing a typical orbital velocity and  $c$  standing for the speed of light in vacuum. For this reason, they are often referred to as the 2.5 post-Newtonian (PN) damping. The measurements resulting from the observation of the Hulse-Taylor binary pulsar PSR 1913+16 show excellent agreement with the values predicted by Einstein theory [5].

A while ago, Iyer and Will [6] derived the equations of motion for two non-spinning point-like objects including the first dissipative terms appearing at the 2.5 and 3.5PN level, *i.e.* the quadrupole damping and its 1PN correction. They proceeded by postulating a balance equation between the instantaneous flux of energy and angular momentum in the far zone on the one hand, and the instantaneous loss of energy and angular momentum in the

system's near zone on the other hand. In a later paper [2], they detailed their derivation and proposed an alternative method, based on the specialization of the Blanchet's radiation-reaction potentials to binary systems, which led to the same results [7]. More recently, the dynamics of a two-body system at the 3.5PN order was derived by Pati and Will in harmonic coordinates [8] using the integration method for the Einstein field equations developed in Ref. [9].

This article deals with the 3.5PN gravitational damping of compact binaries regarded as point-masses. We base our investigation on the  $N$ -body point-mass Hamiltonian [1] calculated by Jaranowski and Schäfer within the Arnowitt-Deser-Misner (ADM) Hamiltonian formalism of general relativity [10], which has proved to be very efficient in determining the approximate 3.5PN dynamics [1, 11, 12, 13]. A remarkable feature of the ADM formalism at this level is the absence of asymptotic matching between the near field and the far field given in a near-zone and a far-zone-defined coordinate system respectively. We can stay in one single global coordinate system in all our calculations, like in Ref. [8]. Let us mention that the ADM formalism does not provide directly the balance equations between instantaneous losses and fluxes, even when assuming quasi-stationarity in the radiation emission. Although the ADM formalism gives us the conserved total energy, the balance between the lost energy of the matter system and the energy flux in the wave zone is proved up to 3.5PN order only [see, e.g., Ref. [14]].

The main purposes of this work are (i) to specialize the  $N$ -body 3.5PN ADM Hamiltonian of Ref. [1] to  $N = 2$ , (ii) to deduce from it the energy loss of a two-body system in general orbits, and (iii) to calculate the equations of motions in the Hamiltonian form and in the Euler-Lagrangian form in order to determine the radiation-reaction force. The expression obtained for the energy loss is different from that derived in Jaranowski and Schäfer [1] and in Iyer and Will [2], but the time-averaged expressions coincide in the case of quasi-elliptic orbital

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motion. The plan of the paper is to reduce the original Hamiltonian for two point-masses. Next, we perform a partial time differentiation to get the energy loss. On the other side, we compute the radiation-reaction force from the Euler-Lagrangian equations of motion. The corresponding energy loss turns out to be equivalent to the preceding one after averaging. Our results are compared with those of Iyer and Will and the accessory gauge coefficients are determined explicitly. Hadamard regularization is systematically employed.

We use units in which  $16\pi G = 1 = c$ , where  $G$  is the Newtonian gravitational constant, but  $G$  and  $c$  are restored in the presentation of our final results. In our notation,  $t$  is the coordinate time. Characters in bold, like  $\mathbf{x} = (x^i)$ , represents a point in the 3-dimensional Euclidean space  $\mathbb{R}^3$  endowed with a standard Euclidean metric. The scalar product is denoted by a dot. Letters  $a, b, \dots$  are particle labels, so that  $\mathbf{x}_a \in \mathbb{R}^3$  denotes the position of the  $a$ th particle. We also define:  $\mathbf{r}_a := \mathbf{x} - \mathbf{x}_a$ ,  $r_a := |\mathbf{r}_a|$ ,  $\mathbf{n}_a := \mathbf{r}_a/r_a$ ; and for  $a \neq b$ :  $\mathbf{r}_{ab} := \mathbf{x}_a - \mathbf{x}_b = \mathbf{r}_b - \mathbf{r}_a$ ,  $r_{ab} := |\mathbf{r}_{ab}|$ ,  $\mathbf{n}_{ab} := \mathbf{r}_{ab}/r_{ab}$ ;  $|\dots|$  stands here for the Euclidean length of a vector. The momentum vector of the particle  $a$  with mass parameter  $m_a$  is denoted by  $\mathbf{p}_a = (p_{ai})$ . A dot over a symbol, like in  $\dot{\mathbf{x}}_a$ , represents its total time derivative. The partial differentiation with respect to  $x^i$  is denoted by  $\partial_i$ , or equivalently by a comma, *i.e.*  $\partial_i \phi \equiv \phi_{,i}$ .

## II. 3.5PN FIELD EQUATIONS

The ADM formulation is based on a  $(3+1)$  splitting of space-time. The canonical variables are, on one hand, the projection  $g_{ij}$  of the 3-metric on the space-like hypersurfaces  $x^0 = t = \text{const.}$  in a suitable coordinate grid and, on the other hand, its conjugate momentum  $\pi^{ij}$ . In the so-called ADM gauge [11, 12, 15], the spatial metric  $g_{ij}$  decomposes into a diagonal part  $(1 + \frac{1}{8}\phi)^4 \delta_{ij}$ , plus a part  $h_{ij}^{\text{TT}}$  transverse and traceless with respect to the Euclidean metric  $\delta_{ij}$ , so that

$$g_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\text{TT}} \quad (2.1)$$

with  $\partial_j h_{ij}^{\text{TT}} = 0$ . In addition, the field momentum  $\pi^{ij}$  is traceless:  $\pi^{ii} = 0$ . Let us consider now  $h_{ij}^{\text{TT}}$  as the new field variable, and denote its conjugate momentum by  $\pi^{ij\text{TT}}$ . The degrees of freedom associated with  $\tilde{\pi}^{ij} = \pi^{ij} - \pi^{ij\text{TT}}$  and the potential  $\phi$  are removed by solving the constraint equations.

After inserting the values of  $\tilde{\pi}^{ij}$  and  $\phi$  into the general relativistic Hamiltonian of the isolated system under consideration in an asymptotically flat space-time, the latter simplifies to a surface integral often referred to as the reduced Hamiltonian  $H$ . It is a functional of the independent degrees of freedom  $h_{ij}^{\text{TT}}$  and  $\pi^{ij\text{TT}}$  [10].

For a system of  $N$  point-like bodies with position vectors  $\mathbf{x}_a$  and momenta  $\mathbf{p}_a$  ( $a = 1, \dots, N$ ), the Hamiltonian

takes the form

$$H = H[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi^{ij\text{TT}}] \quad (2.2)$$

The Hamiltonian equations of motion for body  $a$  are

$$\dot{\mathbf{p}}_a = -\frac{\partial H}{\partial \mathbf{x}_a} \quad \text{and} \quad \dot{\mathbf{x}}_a = \frac{\partial H}{\partial \mathbf{p}_a}, \quad (2.3)$$

and the field equations for the independent degrees of freedom read

$$\frac{\partial}{\partial t} \pi^{ij\text{TT}} = -\delta_{kl}^{\text{TT}ij} \frac{\delta H}{\delta h_{kl}^{\text{TT}}}, \quad \frac{\partial}{\partial t} h_{ij}^{\text{TT}} = \delta_{ij}^{\text{TT}kl} \frac{\delta H}{\delta \pi^{kl\text{TT}}}, \quad (2.4)$$

where  $(\delta \dots)/(\delta \dots)$  denotes the Fréchet derivative [see, e.g., Ref. [16]] and where  $\delta_{kl}^{\text{TT}ij}$  is the TT-projection operator [see, e.g., Eq. (2.17) of [12]].

The Hamiltonian  $H$  can be uniquely decomposed into a matter part,  $H^{\text{mat}}$ , depending on the matter variables only, a field part,  $H^{\text{field}}$ , depending on the field variables only, and an interaction part,  $H^{\text{int}}$ , depending on both sets of variables, matter and field, in a way that it vanishes if one set is put to zero. In short, the full content of the field-plus-matter dynamics at the 3.5PN order is included in the Hamiltonian

$$H_{\leq 3.5\text{PN}} = H_{\leq 3.5\text{PN}}^{\text{mat}} + H_{\leq 3.5\text{PN}}^{\text{field}} + H_{\leq 3.5\text{PN}}^{\text{int}}. \quad (2.5)$$

The notation  $\leq 3.5\text{PN}$  indicates that all orders lower or equal than 3.5PN are included. In what follows, only the interaction part of the Hamiltonian (2.5) will be needed [12]. It has been explicitly computed at the dissipative 3.5PN order in Ref. [1] by Jaranowski and Schäfer. Its truncation at the 2.5PN level,  $H_{2.5\text{PN}}^{\text{int}}$ , takes the well-known form presented in Refs. [12, 17]

$$H_{2.5\text{PN}}^{\text{int}}(\mathbf{x}_a, \mathbf{p}_a, t) = 5\pi \dot{\chi}_{(4)ij}(t) \chi_{(4)ij}(\mathbf{x}_a, \mathbf{p}_a) \quad (2.6)$$

with

$$\begin{aligned} \chi_{(4)ij}(\mathbf{x}_a, \mathbf{p}_a) := & \frac{1}{60\pi} \left[ \sum_a \frac{2}{m_a} (\mathbf{p}_a^2 \delta_{ij} - 3p_{ai}p_{aj}) \right. \\ & \left. + \frac{1}{16\pi} \sum_a \sum_{b \neq a} \frac{m_a m_b}{r_{ab}} (3n_{ab}^i n_{ab}^j - \delta_{ij}) \right]. \end{aligned} \quad (2.7)$$

The function  $\dot{\chi}_{(4)ij}(t)$  comes from the post-Newtonian expansion of the retarded integral giving  $h_{ij}^{\text{TT}}$  as a function of  $\mathbf{x}_a$  and  $\mathbf{p}_a$  [1]. As  $\partial H/\partial \mathbf{x}_a$  and  $\partial H/\partial \mathbf{p}_a$  are differentiated for constant  $h_{ij}^{\text{TT}}$  and  $\pi^{ij\text{TT}}$ , the positions and momenta appearing in Eq. (2.7) are not affected by the operators  $\partial/\partial \mathbf{x}_a$  and  $\partial/\partial \mathbf{p}_a$ . In order not to confuse the latter particle variables with the ones that are already present in the original Hamiltonian (2.2) and are affected by  $\partial/\partial \mathbf{x}_a$  and  $\partial/\partial \mathbf{p}_a$ , we shall mark them with a prime symbol:  $\mathbf{x}_{a'}$ ,  $\mathbf{p}_{a'}$ . Thus, in our notation,  $\dot{\chi}_{(4)ij}(t)$  denotes the time derivative of  $\chi_{(4)ij}(\mathbf{x}_{a'}, \mathbf{p}_{a'})$ .

The 3.5PN Hamiltonian  $H_{3.5\text{PN}}^{\text{int}}$  reads<sup>1</sup>

$$H_{3.5\text{PN}}^{\text{int}}(\mathbf{x}_a, \mathbf{p}_a, t) = 5\pi \chi_{(4)ij}(\mathbf{x}_a, \mathbf{p}_a) \left[ \dot{\Pi}_{1ij}(t) + \dot{\Pi}_{2ij}(t) + \boxed{\ddot{\Pi}_{3ij}(t)} \right] + 5\pi \boxed{\dot{\chi}_{(4)ij}(t)} \left[ \Pi_{1ij}(\mathbf{x}_a, \mathbf{p}_a) + \tilde{\Pi}_{2ij}(\mathbf{x}_a, t) \right] \\ - 5\pi \ddot{\chi}_{(4)ij}(t) \Pi_{3ij}(\mathbf{x}_a, \mathbf{p}_a) + \dot{\chi}_{(4)ij}(t) \left[ Q'_{ij}(\mathbf{x}_a, \mathbf{p}_a, t) + Q''_{ij}(\mathbf{x}_a, t) \right] \\ + \frac{\partial^3}{\partial t^3} \left[ R'(\mathbf{x}_a, \mathbf{p}_a, t) + R''(\mathbf{x}_a, t) \right]. \quad (2.8)$$

$H_{3.5\text{PN}}^{\text{int}}$  is parametrized by the following functions

$$\Pi_{1ij}(\mathbf{x}_a, \mathbf{p}_a) := \frac{4}{15} \left( \frac{1}{16\pi} \right) \sum_a \frac{\mathbf{p}_a^2}{m_a^3} (-\mathbf{p}_a^2 \delta_{ij} + 3p_{ai}p_{aj}) + \frac{8}{5} \left( \frac{1}{16\pi} \right)^2 \sum_a \sum_{b \neq a} \frac{m_b}{m_a r_{ab}} (-2\mathbf{p}_a^2 \delta_{ij} + 5p_{ai}p_{aj} + \mathbf{p}_a^2 n_{ab}^i n_{ab}^j) \\ + \frac{1}{5} \left( \frac{1}{16\pi} \right)^2 \sum_a \sum_{b \neq a} \frac{1}{r_{ab}} \left\{ \left[ 19(\mathbf{p}_a \cdot \mathbf{p}_b) - 3(\mathbf{n}_{ab} \cdot \mathbf{p}_a)(\mathbf{n}_{ab} \cdot \mathbf{p}_b) \right] \delta_{ij} - 42p_{ai}p_{bj} \right. \\ \left. - 3 \left[ 5(\mathbf{p}_a \cdot \mathbf{p}_b) + (\mathbf{n}_{ab} \cdot \mathbf{p}_a)(\mathbf{n}_{ab} \cdot \mathbf{p}_b) \right] n_{ab}^i n_{ab}^j + 6(\mathbf{n}_{ab} \cdot \mathbf{p}_b) (n_{ab}^i p_{aj} + n_{ab}^j p_{ai}) \right\} \\ + \frac{41}{15} \left( \frac{1}{16\pi} \right)^3 \sum_a \sum_{b \neq a} \frac{m_a^2 m_b}{r_{ab}^2} (\delta_{ij} - 3n_{ab}^i n_{ab}^j) \\ + \frac{1}{45} \left( \frac{1}{16\pi} \right)^3 \sum_a \sum_{b \neq a} \sum_{c \neq a, b} m_a m_b m_c \left\{ \frac{18}{r_{ab} r_{ca}} (\delta_{ij} - 3n_{ab}^i n_{ab}^j) \right. \\ \left. - \frac{180}{s_{abc}} \left[ \left( \frac{1}{r_{ab}} + \frac{1}{s_{abc}} \right) n_{ab}^i n_{ab}^j + \frac{1}{s_{abc}} n_{ab}^i n_{bc}^j \right] \right. \\ \left. + \frac{10}{s_{abc}} \left[ 4 \left( \frac{1}{r_{ab}} + \frac{1}{r_{bc}} + \frac{1}{r_{ca}} \right) - \frac{r_{ab}^2 + r_{bc}^2 + r_{ca}^2}{r_{ab} r_{bc} r_{ca}} \right] \delta_{ij} \right\}, \quad (2.9a)$$

with  $s_{abc} := r_{ab} + r_{bc} + r_{ca}$ ,

$$\Pi_{2ij}(\mathbf{x}_a, \mathbf{p}_a) := \frac{1}{5} \left( \frac{1}{16\pi} \right)^2 \sum_a \sum_{b \neq a} \frac{m_b}{m_a r_{ab}} \left\{ \left[ 5(\mathbf{n}_{ab} \cdot \mathbf{p}_a)^2 - \mathbf{p}_a^2 \right] \delta_{ij} - 2p_{ai}p_{aj} + \left[ 5\mathbf{p}_a^2 - 3(\mathbf{n}_{ab} \cdot \mathbf{p}_a)^2 \right] n_{ab}^i n_{ab}^j \right. \\ \left. - 6(\mathbf{n}_{ab} \cdot \mathbf{p}_a)(n_{ab}^i p_{aj} + n_{ab}^j p_{ai}) \right\} + \frac{6}{5} \left( \frac{1}{16\pi} \right)^3 \sum_a \sum_{b \neq a} \frac{m_a^2 m_b}{r_{ab}^2} (3n_{ab}^i n_{ab}^j - \delta_{ij}) \\ + \frac{1}{10} \left( \frac{1}{16\pi} \right)^3 \sum_a \sum_{b \neq a} \sum_{c \neq a, b} m_a m_b m_c \left\{ \left[ 5 \frac{r_{ca}}{r_{ab}^3} \left( 1 - \frac{r_{ca}}{r_{bc}} \right) + \frac{13}{r_{ab} r_{ca}} - \frac{40}{r_{ab} s_{abc}} \right] \delta_{ij} \right. \\ + \left[ 3 \frac{r_{ab}}{r_{ca}^3} + \frac{r_{bc}^2}{r_{ab} r_{ca}^3} - \frac{5}{r_{ab} r_{ca}} + \frac{40}{s_{abc}} \left( \frac{1}{r_{ab}} + \frac{1}{s_{abc}} \right) \right] n_{ab}^i n_{ab}^j \\ \left. + \left[ 2 \frac{(r_{ab} + r_{ca})}{r_{bc}^3} - 16 \left( \frac{1}{r_{ab}^2} + \frac{1}{r_{ca}^2} \right) + \frac{88}{s_{abc}^2} \right] n_{ab}^i n_{ca}^j \right\}, \quad (2.9b)$$

$$\Pi_{3ij}(\mathbf{x}_a, \mathbf{p}_a) := \frac{1}{5} \left( \frac{1}{16\pi} \right)^2 \sum_a \sum_{b \neq a} m_b \left[ -5(\mathbf{n}_{ab} \cdot \mathbf{p}_a) \delta_{ij} + (\mathbf{n}_{ab} \cdot \mathbf{p}_a) n_{ab}^i n_{ab}^j + 7(n_{ab}^i p_{aj} + n_{ab}^j p_{ai}) \right], \quad (2.9c)$$

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<sup>1</sup> We observe that there are few misprints in the expressions for  $H_{3.5\text{PN}}^{\text{int}}$ , appearing in [1]. We make appropriate changes in those expressions and their positions are marked by  $\boxed{\dots}$ .

$$\begin{aligned}
\tilde{\Pi}_{2ij}(\mathbf{x}_a, t) := & \frac{1}{5} \left( \frac{1}{16\pi} \right)^2 \sum_a \sum_{a'} \frac{m_a}{m_{a'} r_{aa'}} \left\{ \left[ 5(\mathbf{n}_{aa'} \cdot \mathbf{p}_{a'})^2 - \mathbf{p}_{a'}^2 \right] \delta_{ij} - 2p_{a'i} p_{a'j} \right. \\
& + \left[ 5\mathbf{p}_{a'}^2 - 3(\mathbf{n}_{aa'} \cdot \mathbf{p}_{a'})^2 \right] n_{aa'}^i n_{aa'}^j - 6(\mathbf{n}_{aa'} \cdot \mathbf{p}_{a'}) \left( n_{aa'}^i p_{a'j} + n_{aa'}^j p_{a'i} \right) \Big\} \\
& + \frac{1}{10} \left( \frac{1}{16\pi} \right)^3 \sum_a \sum_{a'} \sum_{b' \neq a'} m_a m_{a'} m_{b'} \left\{ \frac{32}{s_{aa'b'}} \left( \frac{1}{r_{a'b'}} + \frac{1}{s_{aa'b'}} \right) n_{a'b'}^i n_{a'b'}^j \right. \\
& + 16 \left( \frac{1}{r_{a'b'}^2} - \frac{2}{s_{aa'b'}^2} \right) \left( n_{aa'}^i n_{a'b'}^j + n_{aa'}^j n_{a'b'}^i \right) - 2 \left( \frac{r_{aa'} + r_{ab'}}{r_{a'b'}^3} + \frac{12}{s_{aa'b'}^2} \right) n_{aa'}^i n_{ab'}^j \\
& + \left[ \frac{r_{aa'}}{r_{a'b'}^3} \left( \frac{r_{aa'}}{r_{ab'}} + 3 \right) - \frac{5}{r_{a'b'} r_{aa'}} + \frac{8}{s_{aa'b'}} \left( \frac{1}{r_{aa'}} + \frac{1}{s_{aa'b'}} \right) \right] n_{aa'}^i n_{aa'}^j \\
& \left. + \left[ 5 \frac{r_{aa'}}{r_{a'b'}^3} \left( 1 - \frac{r_{aa'}}{r_{ab'}} \right) + \frac{17}{r_{a'b'} r_{aa'}} - \frac{4}{r_{aa'} r_{ab'}} - \frac{8}{s_{aa'b'}} \left( \frac{1}{r_{aa'}} + \frac{4}{r_{a'b'}} \right) \right] \delta_{ij} \right\}, \quad (2.9d)
\end{aligned}$$

with  $s_{aa'b'} := r_{aa'} + r_{ab'} + r_{a'b'}$ . Variables originating from  $h_{ij}^{\text{TT}}$  and  $\pi^{ij\text{TT}}$  are primed. It is indeed crucial to distinguish the particle positions and momenta inside and outside the transverse-traceless (TT) quantities at this stage.

The other four functions which enter  $H_{3.5\text{PN}}^{\text{int}}$  are

$$Q'_{ij}(\mathbf{x}_a, \mathbf{p}_a, t) := -\frac{1}{16} \left( \frac{1}{16\pi} \right) \sum_a \sum_{a'} \frac{m_a}{m_a r_{aa'}} \left[ 2p_{ai} p_{aj} + 12(\mathbf{n}_{aa'} \cdot \mathbf{p}_a) n_{aa'}^i p_{aj} - 5\mathbf{p}_a^2 n_{aa'}^i n_{aa'}^j + 3(\mathbf{n}_{aa'} \cdot \mathbf{p}_a)^2 n_{aa'}^i n_{aa'}^j \right], \quad (2.9e)$$

$$\begin{aligned}
Q''_{ij}(\mathbf{x}_a, t) := & \frac{1}{32} \left( \frac{1}{16\pi} \right)^2 \sum_a \sum_{b \neq a} \sum_{a'} m_a m_b m_{a'} \left\{ \frac{32}{s_{aba'}} \left( \frac{1}{r_{ab}} + \frac{1}{s_{aba'}} \right) n_{ab}^i n_{ab}^j + \left[ 3 \frac{r_{aa'}}{r_{ab}^3} - \frac{5}{r_{ab} r_{aa'}} + \frac{r_{ba'}^2}{r_{ab}^3 r_{aa'}} \right. \right. \\
& \left. \left. + \frac{8}{s_{aba'}} \left( \frac{1}{r_{aa'}} + \frac{1}{s_{aba'}} \right) \right] n_{aa'}^i n_{aa'}^j - 2 \left( \frac{r_{aa'} + r_{ba'}}{r_{ab}^3} + \frac{12}{s_{aba'}^2} \right) n_{aa'}^i n_{ba'}^j - 32 \left( \frac{1}{r_{ab}^2} - \frac{2}{s_{aba'}^2} \right) n_{ab}^i n_{aa'}^j \right\}, \quad (2.9f)
\end{aligned}$$

with  $s_{aba'} := r_{ab} + r_{aa'} + r_{ba'}$ ,

$$\begin{aligned}
R'(\mathbf{x}_a, \mathbf{p}_a, t) := & \frac{2}{105} \left( \frac{1}{16\pi} \right) \sum_a \sum_{a'} \frac{r_{aa'}^2}{m_a m_{a'}} \left[ -5\mathbf{p}_a^2 \mathbf{p}_{a'}^2 + 11(\mathbf{p}_a \cdot \mathbf{p}_{a'})^2 + 4(\mathbf{n}_{aa'} \cdot \mathbf{p}_{a'})^2 \mathbf{p}_a^2 \right. \\
& \left. + 4(\mathbf{n}_{aa'} \cdot \mathbf{p}_a)^2 \mathbf{p}_{a'}^2 - 12(\mathbf{n}_{aa'} \cdot \mathbf{p}_{a'}) (\mathbf{n}_{aa'} \cdot \mathbf{p}_a) (\mathbf{p}_a \cdot \mathbf{p}_{a'}) \right] \\
& - \frac{1}{105} \left( \frac{1}{16\pi} \right)^2 \sum_a \sum_{a'} \sum_{b' \neq a'} \frac{m_{a'} m_{b'}}{m_a} \left[ \left( 2 \frac{r_{aa'}^4}{r_{a'b'}^3} - 2 \frac{r_{aa'}^2 r_{ab'}^2}{r_{a'b'}^3} - 5 \frac{r_{aa'}^2}{r_{a'b'}} \right) \mathbf{p}_a^2 + 4 \frac{r_{aa'}^2}{r_{a'b'}} (\mathbf{n}_{aa'} \cdot \mathbf{p}_a)^2 \right. \\
& \left. + 17 \left( \frac{r_{aa'}^2}{r_{a'b'}} + r_{a'b'} \right) (\mathbf{n}_{a'b'} \cdot \mathbf{p}_a)^2 + 2 \left( 6 \frac{r_{aa'}^3}{r_{a'b'}^2} + 17 r_{aa'} \right) (\mathbf{n}_{aa'} \cdot \mathbf{p}_a) (\mathbf{n}_{a'b'} \cdot \mathbf{p}_a) \right] \quad (2.9g)
\end{aligned}$$

and

$$\begin{aligned}
R''(\mathbf{x}_a, t) := & \frac{1}{105} \left( \frac{1}{16\pi} \right)^2 \sum_a \sum_{b \neq a} \sum_{a'} \frac{m_a m_b}{m_{a'}} \left[ \left( 5 \frac{r_{aa'}^2}{r_{ab}} + 2 \frac{r_{aa'}^2 r_{ba'}^2}{r_{ab}^3} - 2 \frac{r_{aa'}^4}{r_{ab}^3} \right) \mathbf{p}_{a'}^2 - 17 \left( \frac{r_{aa'}^2}{r_{ab}} + r_{ab} \right) (\mathbf{n}_{ab} \cdot \mathbf{p}_{a'})^2 \right. \\
& \left. - 4 \frac{r_{aa'}^2}{r_{ab}} (\mathbf{n}_{aa'} \cdot \mathbf{p}_{a'})^2 - 2 \left( \frac{6 r_{aa'}^3}{r_{ab}^2} + 17 r_{aa'} \right) (\mathbf{n}_{ab} \cdot \mathbf{p}_{a'}) (\mathbf{n}_{aa'} \cdot \mathbf{p}_{a'}) \right] \\
& + \frac{1}{210} \left( \frac{1}{16\pi} \right)^3 \sum_a \sum_{b \neq a} \sum_{a'} \sum_{b' \neq a'} m_a m_b m_{a'} m_{b'} \left[ 2 \frac{r_{aa'}^2}{r_{ab} r_{a'b'}^3} (r_{aa'}^2 - r_{ab'}^2) + 2 \frac{r_{aa'}^2}{r_{ab}^3 r_{a'b'}} (r_{aa'}^2 - r_{ba'}^2) \right. \\
& \left. + 2 \frac{r_{aa'}^2}{r_{ab}^3 r_{a'b'}} (r_{aa'}^2 - r_{ab'}^2) + 2 \frac{r_{aa'}^2}{r_{ab}^3 r_{a'b'}} (r_{aa'}^2 - r_{ba'}^2) \right]
\end{aligned}$$

$$\begin{aligned}
& + 4 \frac{r_{ab} r_{aa'}^2}{r_{a'b'}^3} - 5 \frac{r_{aa'}^2}{r_{ab} r_{a'b'}} - 2 \left( \frac{r_{ab}^3}{r_{a'b'}^3} + \frac{r_{ab}}{r_{a'b'}} \right) + 17 \left( \frac{r_{ab}}{r_{a'b'}} + \frac{r_{a'b'}}{r_{ab}} + \frac{r_{aa'}^2}{r_{ab} r_{a'b'}} \right) (\mathbf{n}_{ab} \cdot \mathbf{n}_{a'b'})^2 \\
& - 4 \frac{r_{ab} r_{aa'} r_{bb'}}{r_{a'b'}^3} (\mathbf{n}_{aa'} \cdot \mathbf{n}_{bb'}) + 6 \frac{r_{aa'}^4}{r_{ab}^2 r_{a'b'}^2} (\mathbf{n}_{ab} \cdot \mathbf{n}_{a'b'}) + 34 r_{aa'}^2 \left( \frac{1}{r_{ab}^2} + \frac{1}{r_{a'b'}^2} \right) (\mathbf{n}_{ab} \cdot \mathbf{n}_{a'b'}) \Big]. \quad (2.9h)
\end{aligned}$$

Now, we are in a position to reduce the  $N$ -body 3.5PN Hamiltonian (2.8) to  $N = 2$ . The result will serve as a base for the derivation of the gravitational energy loss of a compact binary moving in general orbits, like in Ref. [1], and provide a reference to check the equations of motion that will be calculated afterward from this two-body Hamiltonian.

### III. ENERGY LOSS OF A TWO-BODY SYSTEM

Let us denote by  $\tilde{H}_{\leq 3.5\text{PN}}$  the Hamiltonian resulting from  $H_{\leq 3.5\text{PN}}$ , Eq. (2.5) after writing the field part as a function of  $\mathbf{x}_a, \mathbf{p}_a, \dot{\mathbf{x}}_a, \dot{\mathbf{p}}_a$  and subtracting its 2.5PN and 3.5PN contribution  $H_{2.5\text{PN}}^{\text{field}}$  and  $H_{3.5\text{PN}}^{\text{field}}$  from  $H_{\leq 3.5\text{PN}}^{\text{field}}$ . By definition, we have

$$\begin{aligned}
\tilde{H}_{\leq 3.5\text{PN}} &:= H_{\leq 3.5\text{PN}}^{\text{mat}} + H_{\leq 3.5\text{PN}}^{\text{field}} - H_{2.5\text{PN}}^{\text{field}} - H_{3.5\text{PN}}^{\text{field}} \\
&+ H_{\leq 3.5\text{PN}}^{\text{int}}, \quad (3.1)
\end{aligned}$$

which can be split into a conservative and a dissipative part

$$\begin{aligned}
\tilde{H}_{\leq 3.5\text{PN}}(\mathbf{x}_a, \mathbf{p}_a, \dot{\mathbf{x}}_a, \dot{\mathbf{p}}_a, t) &= H_{\leq 3\text{PN}}^{\text{cons}}(\mathbf{x}_a, \mathbf{p}_a, \dot{\mathbf{x}}_a, \dot{\mathbf{p}}_a) \\
&+ H_{\leq 3.5\text{PN}}^{\text{diss}}(\mathbf{x}_a, \mathbf{p}_a, t), \quad (3.2)
\end{aligned}$$

where

$$\begin{aligned}
H_{\leq 3\text{PN}}^{\text{cons}} &:= H_N^{\text{mat}} + H_{1\text{PN}}^{\text{mat}} + (H_{2\text{PN}}^{\text{mat}} + H_{2\text{PN}}^{\text{field}} + H_{2\text{PN}}^{\text{int}}) \\
&+ (H_{3\text{PN}}^{\text{mat}} + H_{3\text{PN}}^{\text{field}} + H_{3\text{PN}}^{\text{int}}), \quad (3.3a)
\end{aligned}$$

$$H_{\leq 3.5\text{PN}}^{\text{diss}} := H_{2.5\text{PN}}^{\text{int}} + H_{3.5\text{PN}}^{\text{int}}. \quad (3.3b)$$

The conservative Hamilton function corresponding to the expression (3.3a) is given in Refs. [18, 19]. With the help of the coordinate transformation described in Ref. [13], it is possible to eliminate the dependence on  $\dot{\mathbf{x}}_a$  and  $\dot{\mathbf{p}}_a$ . The difference between  $H_{\leq 3\text{PN}}^{\text{cons}}$  and  $H_{\leq 3.5\text{PN}}^{\text{diss}}$  in Eq. (3.2)

<sup>2</sup> All algebraic computations are performed using the commercial software Mathematica [20].

relates to their behaviors with respect to time reversal:  $H_{\leq 3\text{PN}}^{\text{cons}}$  is symmetric whereas  $H_{\leq 3.5\text{PN}}^{\text{diss}}$  is antisymmetric.

The total time derivative of  $\tilde{H}_{\leq 3.5\text{PN}}$  is equal to its partial time derivative, the other terms canceling by virtue of the equations of motion. Only the dissipative part of  $\tilde{H}_{\leq 3.5\text{PN}}$  depends explicitly on time, through the substitution of the transverse-traceless variables that have been replaced by their actual values. We get therefore

$$\begin{aligned}
\frac{d}{dt} \tilde{H}_{\leq 3.5\text{PN}}(\mathbf{x}_a, \mathbf{p}_a, \dot{\mathbf{x}}_a, \dot{\mathbf{p}}_a, t) &= \frac{\partial}{\partial t} \tilde{H}_{\leq 3.5\text{PN}}(\mathbf{x}_a, \mathbf{p}_a, \dot{\mathbf{x}}_a, \dot{\mathbf{p}}_a, t) \\
&= \frac{\partial}{\partial t} H_{\leq 3.5\text{PN}}^{\text{diss}}(\mathbf{x}_a, \mathbf{p}_a, t). \quad (3.4)
\end{aligned}$$

The instantaneous energy loss of the matter system due to the gravitational wave emission is defined as

$$\mathcal{L}_{\leq 3.5\text{PN}}(t) := -\frac{d}{dt} \tilde{H}_{\leq 3.5\text{PN}}(\mathbf{x}_a, \mathbf{p}_a, \dot{\mathbf{x}}_a, \dot{\mathbf{p}}_a, t). \quad (3.5)$$

Our task is now to compute  $\mathcal{L}_{\leq 3.5\text{PN}}$  for a system of two point-like bodies with masses  $m_1$  and  $m_2$ .<sup>2</sup> We start by rewriting the right-hand side of Eq. (3.5) [by means of Eqs. (3.3b) and (3.4)] in the form

$$\mathcal{L}_{\leq 3.5\text{PN}} = -\frac{\partial}{\partial t} (H_{2.5\text{PN}}^{\text{int}} + H_{3.5\text{PN}}^{\text{int}}). \quad (3.6)$$

Next, we insert the formulas (2.6) and (2.8) into Eq. (3.6), and specialize all sums of Eqs. (2.7) and (2.9) for  $N = 2$ . Notice that the triple sum  $\sum_a \sum_{b \neq a} \sum_{c \neq a, b}$  disappears in rela-

tions (2.9a) and (2.9b). For clarity, we also reintroduce the Newtonian gravitational constant  $G$  and the speed of light  $c$ .

In differentiating the Hamiltonians (2.6) and (2.8) with respect to time, we replace the time derivatives of the primed coordinates and momenta according to the 1PN equations of motion:

$$\dot{\mathbf{x}}_{1'} = \frac{\mathbf{p}_{1'}}{m_{1'}} - \frac{1}{2c^2 m_{1'}^3} \mathbf{p}_{1'}^2 \mathbf{p}_{1'} - \frac{G}{2c^2 r_{1'2'}} \left[ 6 \frac{m_{2'}}{m_{1'}} \mathbf{p}_{1'} - 7 \mathbf{p}_{2'} - (\mathbf{n}_{1'2'} \cdot \mathbf{p}_{2'}) \mathbf{n}_{1'2'} \right], \quad (3.7a)$$

$$\dot{\mathbf{x}}_{2'} = \frac{\mathbf{p}_{2'}}{m_{2'}} - \frac{1}{2c^2 m_{2'}^3} \mathbf{p}_{2'}^2 \mathbf{p}_{2'} - \frac{G}{2c^2 r_{1'2'}} \left[ 6 \frac{m_{1'}}{m_{2'}} \mathbf{p}_{2'} - 7 \mathbf{p}_{1'} - (\mathbf{n}_{1'2'} \cdot \mathbf{p}_{1'}) \mathbf{n}_{1'2'} \right], \quad (3.7b)$$

$$\begin{aligned} \dot{\mathbf{p}}_{1'} = & -\frac{G m_{1'} m_{2'}}{r_{1'2'}^2} \mathbf{n}_{1'2'} + \frac{G^2 m_{1'} m_{2'} (m_{1'} + m_{2'})}{c^2 r_{1'2'}^3} \mathbf{n}_{1'2'} + \frac{G}{2 c^2 r_{1'2'}^2} \left\{ -(\mathbf{n}_{1'2'} \cdot \mathbf{p}_{2'}) \mathbf{p}_{1'} - (\mathbf{n}_{1'2'} \cdot \mathbf{p}_{1'}) \mathbf{p}_{2'} \right. \\ & \left. + \left[ -3 \frac{m_{2'}}{m_{1'}} \mathbf{p}_{1'}^2 - 3 \frac{m_{1'}}{m_{2'}} \mathbf{p}_{2'}^2 + 7 (\mathbf{p}_{1'} \cdot \mathbf{p}_{2'}) + 3 (\mathbf{n}_{1'2'} \cdot \mathbf{p}_{1'}) (\mathbf{n}_{1'2'} \cdot \mathbf{p}_{2'}) \right] \mathbf{n}_{1'2'} \right\}, \end{aligned} \quad (3.7c)$$

$$\begin{aligned} \dot{\mathbf{p}}_{2'} = & \frac{G m_{1'} m_{2'}}{r_{1'2'}^2} \mathbf{n}_{1'2'} - \frac{G^2 m_{1'} m_{2'} (m_{1'} + m_{2'})}{c^2 r_{1'2'}^3} \mathbf{n}_{1'2'} - \frac{G}{2 c^2 r_{1'2'}^2} \left\{ -(\mathbf{n}_{1'2'} \cdot \mathbf{p}_{2'}) \mathbf{p}_{1'} - (\mathbf{n}_{1'2'} \cdot \mathbf{p}_{1'}) \mathbf{p}_{2'} \right. \\ & \left. + \left[ -3 \frac{m_{2'}}{m_{1'}} \mathbf{p}_{1'}^2 - 3 \frac{m_{1'}}{m_{2'}} \mathbf{p}_{2'}^2 + 7 (\mathbf{p}_{1'} \cdot \mathbf{p}_{2'}) + 3 (\mathbf{n}_{1'2'} \cdot \mathbf{p}_{1'}) (\mathbf{n}_{1'2'} \cdot \mathbf{p}_{2'}) \right] \mathbf{n}_{1'2'} \right\}. \end{aligned} \quad (3.7d)$$

The calculation of the third time derivative of  $R'$  and  $R''$  is particularly tedious.

The particle momenta  $\mathbf{p}_{a/a'}$  is expressed in terms of the particle coordinate velocities  $\mathbf{v}_{a/a'}$  up to the 1PN order [see, e.g., Eq. (4.1) of [21]] by:

$$\mathbf{p}_1 = m_1 \mathbf{v}_1 + \frac{1}{2c^2} m_1 \mathbf{v}_1^2 \mathbf{v}_1 + \frac{G m_1 m_2}{2 c^2 r_{12}} \left[ 6 \mathbf{v}_1 - 7 \mathbf{v}_2 - (\mathbf{n}_{12} \cdot \mathbf{v}_2) \mathbf{n}_{12} \right], \quad (3.8a)$$

$$\mathbf{p}_2 = m_2 \mathbf{v}_2 + \frac{1}{2c^2} m_2 \mathbf{v}_2^2 \mathbf{v}_2 + \frac{G m_1 m_2}{2 c^2 r_{12}} \left[ 6 \mathbf{v}_2 - 7 \mathbf{v}_1 - (\mathbf{n}_{12} \cdot \mathbf{v}_1) \mathbf{n}_{12} \right]. \quad (3.8b)$$

Analogous relations hold for  $\mathbf{p}_{1'}$  and  $\mathbf{p}_{2'}$ .

The time derivatives of the primed coordinates  $\mathbf{x}_{1'}$  and  $\mathbf{x}_{2'}$  coming from the differentiation of  $\tilde{H}_{\leq 3.5\text{PN}}$  in Eq. (3.5) equate the primed coordinate velocities  $\mathbf{v}_{1'}$  and  $\mathbf{v}_{2'}$ . We eliminate all accelerations by making use of the 1PN equations of motion [see, e.g., Eq. (1.5) of [15]]:

$$\begin{aligned} \dot{\mathbf{v}}_{1'} = & -\frac{G m_{2'}}{r_{1'2'}^2} \mathbf{n}_{1'2'} + \frac{G m_{2'}}{c^2 r_{1'2'}^2} \left\{ \left[ 4 (\mathbf{n}_{1'2'} \cdot \mathbf{v}_{1'}) - 3 (\mathbf{n}_{1'2'} \cdot \mathbf{v}_{2'}) \right] (\mathbf{v}_{1'} - \mathbf{v}_{2'}) + \left[ -\mathbf{v}_{1'}^2 - 2 \mathbf{v}_{2'}^2 + 4 (\mathbf{v}_{1'} \cdot \mathbf{v}_{2'}) \right. \right. \\ & \left. \left. + \frac{3}{2} (\mathbf{n}_{1'2'} \cdot \mathbf{v}_{2'})^2 + \frac{G}{r_{1'2'}} (5m_{1'} + 4m_{2'}) \right] \mathbf{n}_{12} \right\}, \end{aligned} \quad (3.9a)$$

$$\begin{aligned} \dot{\mathbf{v}}_{2'} = & +\frac{G m_{1'}}{r_{1'2'}^2} \mathbf{n}_{1'2'} + \frac{G m_{1'}}{c^2 r_{1'2'}^2} \left\{ \left[ 4 (\mathbf{n}_{1'2'} \cdot \mathbf{v}_{2'}) - 3 (\mathbf{n}_{1'2'} \cdot \mathbf{v}_{1'}) \right] (\mathbf{v}_{1'} - \mathbf{v}_{2'}) - \left[ -\mathbf{v}_{2'}^2 - 2 \mathbf{v}_{1'}^2 + 4 (\mathbf{v}_{1'} \cdot \mathbf{v}_{2'}) \right. \right. \\ & \left. \left. + \frac{3}{2} (\mathbf{n}_{1'2'} \cdot \mathbf{v}_{1'})^2 + \frac{G}{r_{1'2'}} (5m_{2'} + 4m_{1'}) \right] \mathbf{n}_{1'2'} \right\}. \end{aligned} \quad (3.9b)$$

At this stage, we can identify positions and velocities of particles inside and outside the transverse-traceless variables [*i.e.* the primed and unprimed quantities]. As the limit  $\mathbf{x}_{1'} \rightarrow \mathbf{x}_1$ ,  $\mathbf{x}_{2'} \rightarrow \mathbf{x}_2$  is singular, we shall resort to the Hadamard regularization procedure described in Appendix A [for a more detailed investigation, see paper [22] and the references therein]. The computations are extremely long, but straightforward.

To write the final formula for the energy loss in a more compact way, we introduce the total mass of the system  $M := m_1 + m_2$ , the reduced mass  $\mu := m_1 m_2 / M$ , as well as the ratio parameter  $\nu := \mu / M$ . The individual masses  $m_1$  and  $m_2$  are given in terms of  $\mu$  and  $\nu$  by:

$$m_1 = \frac{\mu}{2\nu} \left( 1 + \sqrt{1 - 4\nu} \right), \quad (3.10a)$$

$$m_2 = \frac{\mu}{2\nu} \left( 1 - \sqrt{1 - 4\nu} \right), \quad (3.10b)$$

assuming  $m_1 \geq m_2$ . We also express the velocities of the bodies in the center-of-mass frame as a function of their relative velocity  $\mathbf{v} = \mathbf{v}_{12} := \mathbf{v}_1 - \mathbf{v}_2$  and of their separation  $\mathbf{r} = \mathbf{r}_{12} := \mathbf{x}_1 - \mathbf{x}_2 = \mathbf{r}_2 - \mathbf{r}_1$  with  $\mathbf{v} = \dot{\mathbf{r}}$ . This is achieved by means of the relations linking the positions of the individual particles to  $\mathbf{r}$  at the 1PN order [see, e.g., Eq. (2.4) of [23]].<sup>3</sup>

$$\mathbf{x}_1 = \left[ \frac{\mu}{m_1} + \frac{\mu(m_1 - m_2)}{2c^2 M^2} \left( \mathbf{v}^2 - \frac{GM}{r} \right) \right] \mathbf{r}, \quad (3.11a)$$

<sup>3</sup> At the 1PN order the Eqs. (3.11) in ADM gauge and in harmonic gauge are identical.

$$\mathbf{x}_2 = \left[ -\frac{\mu}{m_2} + \frac{\mu(m_1 - m_2)}{2c^2 M^2} \left( \mathbf{v}^2 - \frac{GM}{r} \right) \right] \mathbf{r}, \quad (3.11b)$$

where  $r = |\mathbf{r}|$ . By differentiating Eqs. (3.11) with respect to time and eliminating the temporal derivatives with the aid of the Newtonian equation of motion

$$\dot{\mathbf{v}} = -\frac{GM}{r^2} \mathbf{n}, \quad (3.12)$$

where  $\mathbf{n} := \mathbf{r}/r$ , we are led to:

$$\mathbf{v}_1 = \frac{2\nu\mathbf{v}}{1 + \sqrt{1 - 4\nu}} + \frac{\sqrt{1 - 4\nu}}{2c^2} \times$$

$$\times \left[ \nu \mathbf{v}^2 \mathbf{v} - \frac{G\mu}{r} (\mathbf{v} + (\mathbf{n} \cdot \mathbf{v}) \mathbf{n}) \right] \quad (3.13a)$$

$$\mathbf{v}_2 = \frac{-2\nu\mathbf{v}}{1 - \sqrt{1 - 4\nu}} + \frac{\sqrt{1 - 4\nu}}{2c^2} \times$$

$$\times \left[ \nu \mathbf{v}^2 \mathbf{v} - \frac{G\mu}{r} (\mathbf{v} + (\mathbf{n} \cdot \mathbf{v}) \mathbf{n}) \right]. \quad (3.13b)$$

Finally, we arrive at the instantaneous energy loss [24]:

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$$\begin{aligned} \mathcal{L}_{\leq 3.5\text{PN}}^{(1)} = & \frac{4}{15} \frac{G^2 M^3 \nu^2}{c^5 r^3} \left\{ 2 \frac{G^2 M^2}{r^2} + \left[ 11\mathbf{v}^2 - 9(\mathbf{n} \cdot \mathbf{v})^2 \right] \frac{GM}{r} + \left[ 11\mathbf{v}^4 - 60(\mathbf{n} \cdot \mathbf{v})^2 \mathbf{v}^2 + 45(\mathbf{n} \cdot \mathbf{v})^4 \right] \right\} \\ & + \frac{1}{105} \frac{G^2 M^3 \nu^2}{c^7 r^3} \left\{ (5 - 104\nu) \frac{G^3 M^3}{r^3} + \left[ (-1615 + 776\nu)\mathbf{v}^2 + (2733 + 92\nu)(\mathbf{n} \cdot \mathbf{v})^2 \right] \frac{G^2 M^2}{r^2} \right. \\ & + \left[ -(1308 + 1372\nu)\mathbf{v}^4 + 4(4158 + 1577\nu)(\mathbf{n} \cdot \mathbf{v})^2 \mathbf{v}^2 - 3(5174 + 1496\nu)(\mathbf{n} \cdot \mathbf{v})^4 \right] \frac{GM}{r} \\ & + \left[ -(164 + 1064\nu)\mathbf{v}^6 + 3(716 + 1562\nu)(\mathbf{n} \cdot \mathbf{v})^2 \mathbf{v}^4 - 15(324 + 56\nu)(\mathbf{n} \cdot \mathbf{v})^4 \mathbf{v}^2 \right. \\ & \left. \left. + 105(28 - 22\nu)(\mathbf{n} \cdot \mathbf{v})^6 \right] \right\}. \end{aligned} \quad (3.14)$$


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The comparison of the expression above for the instantaneous gravitational energy loss with the known instantaneous far-zone flux [see Eq. (51) of [25] corrected by an erratum and equation below Eq. (3.40) of [26]] shows that they are not either identical. This is nothing to worry about since the form of the instantaneous gravitational energy loss in an isolated system is as well *coordinate* as *representation* dependent.

We also observe that the expression (3.14) is different from formula (73) [in the work of Jaranowski and Schäfer [1]] at the 3.5PN order. However, the difference may be expressed as a total time derivative. To verify that, we computed  $\langle \mathcal{L}_{\leq 3.5\text{PN}}^{(1)} \rangle$  and  $\langle \mathcal{L}_{\leq 3.5\text{PN}}^{\text{JS}} \rangle$ , the orbital average of  $\mathcal{L}_{\leq 3.5\text{PN}}^{(1)}$  and  $\mathcal{L}_{\leq 3.5\text{PN}}^{\text{JS}}$  respectively, using the prescription given in Ref. [26], for quasi-elliptical orbital motion and found  $\langle \mathcal{L}_{\leq 3.5\text{PN}}^{\text{JS}} \rangle = \langle \mathcal{L}_{\leq 3.5\text{PN}}^{(1)} \rangle$ . So, we recover Eqs. (4.20) and (4.21) of Ref. [26]. The important point is that all averaged losses coincide.

In the following, we shall make the assumption that the binary has entered the inspiraling phase, so that its relative motion is quasi-circular. The investigation appears to be simpler in this case, since all terms involving the factor  $(\mathbf{n} \cdot \mathbf{v})$  vanish. The averaging is performed by expressing  $\mathcal{L}_{\leq 3.5\text{PN}}^{(1)}$  as a function of  $E$ . According to equation (3.35) in Ref. [26], we have for the energy

(reduced by the factor  $\mu$ )

$$E = \frac{1}{2} \mathbf{v}^2 - \frac{GM}{r} + \frac{3}{8c^2} (1 - 3\nu) \mathbf{v}^4$$

$$+ \frac{GM}{2rc^2} \left[ (3 + \nu) \mathbf{v}^2 + \nu (\mathbf{n} \cdot \mathbf{v})^2 + \frac{GM}{r} \right]. \quad (3.15)$$

The right-hand side depends on the relative velocity  $\mathbf{v}$  and the relative distance  $r$ , whereas the symbol  $E$  in the left-hand side may be viewed as an adiabatic parameter. By solving Eq. (3.15) for  $\mathbf{v}^2$ , we find

$$\mathbf{v}^2 = 2E + \frac{2GM}{r} + \frac{1}{c^2} \left[ E^2 (-3 + 9\nu) \right.$$

$$\left. + \frac{GM}{r} E (-12 + 16\nu) + \frac{G^2 M^2}{r^2} (-10 + 7\nu) \right]. \quad (3.16)$$

Next, we substitute Eq. (3.16) to  $\mathbf{v}^2$  in the instantaneous energy loss (3.14), and eliminate the relative distance  $r$  in favor of the reduced energy with the aid of Eq. (43) in Ref. [27]:

$$r = -\frac{GM}{2E} \left[ 1 + \frac{1}{2} (7 - \nu) \frac{E}{c^2} \right]. \quad (3.17)$$

As a consequence, the time-averaged energy loss specialized to the case of circular orbits (c.o.) reads

$$\left\langle \mathcal{L}_{\leq 3.5\text{PN}} \right\rangle_t^{\text{c.o.}} = \frac{1024}{5} \frac{\nu^2 (-E)^5}{c^5 G} \left[ 1 + \frac{(-E)}{c^2} \left( \frac{13}{168} - 5\nu \right) \right]. \quad (3.18)$$

We recover formula (4.20) in Ref. [26] with  $e_R = 0$  (due to the fact that the orbits are quasi-circular). This is in agreement with Eq. (81) of Ref. [25]. Processing Eq. (4.6) or Eq. (73) in Ref. [1] in a similar way ends in the same result.

The instantaneous gravitational energy loss  $\mathcal{L}_{\leq 3.5\text{PN}}$  given in Eq. (3.14) will be taken as reference to check the equations of motion that will be determined in the next section.

#### IV. HAMILTONIAN EQUATIONS OF MOTION

In this section we compute the reaction part of the equations of motion

$$\dot{\mathbf{p}}_a = -\frac{\partial H_{\leq 3.5\text{PN}}^{\text{int}}}{\partial \mathbf{x}_a} \quad \text{and} \quad \dot{\mathbf{x}}_a = \frac{\partial H_{\leq 3.5\text{PN}}^{\text{int}}}{\partial \mathbf{p}_a}, \quad (4.1)$$

in a fully explicit form in terms of  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1$  and  $\mathbf{p}_2$ . For this goal, we have to differentiate the Hamiltonians (2.6) and (2.8) with respect to the particle coordinates  $\mathbf{x}_a$  and the particle momenta  $\mathbf{p}_a$  with  $a = 1, 2$  and apply the Hadamard regularization procedure described in Appendix A. We find [24]:

$$\begin{aligned} \dot{\mathbf{x}}_1 = & \frac{1}{c^5} \left\{ \frac{G^2}{r^2} \left\{ \left[ -\frac{24}{5} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) + \frac{m_2}{m_1} \left( \frac{24}{5} (\mathbf{n} \cdot \mathbf{p}_1)^2 - \frac{16}{5} \mathbf{p}_1^2 \right) + \frac{16}{5} (\mathbf{p}_1 \cdot \mathbf{p}_2) \right] \mathbf{n} + \left[ -\frac{8m_2}{3m_1} (\mathbf{n} \cdot \mathbf{p}_1) \right. \right. \right. \\ & \left. \left. - \frac{8}{15} (\mathbf{n} \cdot \mathbf{p}_2) \right] \mathbf{p}_1 + \frac{16}{5} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_2 \right\} + \frac{1}{c^7} \left\{ \frac{G^2}{r^2} \left\{ \left[ \frac{m_2}{m_1^3} \left( \frac{36}{7} (\mathbf{n} \cdot \mathbf{p}_1)^4 - \frac{60}{7} (\mathbf{n} \cdot \mathbf{p}_1)^2 \mathbf{p}_1^2 + \frac{4}{35} \mathbf{p}_1^4 \right) \right. \right. \right. \\ & \left. \left. + \frac{1}{m_1^2} \left( -\frac{148}{7} (\mathbf{n} \cdot \mathbf{p}_1)^3 (\mathbf{n} \cdot \mathbf{p}_2) + \frac{748}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 + \frac{12}{7} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{172}{35} \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) \right. \right. \\ & \left. \left. + \frac{1}{m_1 m_2} \left( \frac{104}{7} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2)^2 - \frac{324}{35} (\mathbf{n} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2 - \frac{432}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) (\mathbf{p}_1 \cdot \mathbf{p}_2) - \frac{136}{35} (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 \right. \right. \right. \\ & \left. \left. + \frac{132}{35} (\mathbf{n} \cdot \mathbf{p}_1)^2 \mathbf{p}_2^2 - \frac{172}{105} \mathbf{p}_1^2 \mathbf{p}_2^2 \right) + \frac{1}{m_2^2} \left( \frac{8}{7} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2)^3 + \frac{292}{35} (\mathbf{n} \cdot \mathbf{p}_2)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) \right. \right. \\ & \left. \left. - \frac{176}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_2^2 + \frac{52}{105} (\mathbf{p}_1 \cdot \mathbf{p}_2) \mathbf{p}_2^2 \right) \right] \mathbf{n} + \left[ \frac{m_2}{m_1^3} \left( -\frac{324}{35} (\mathbf{n} \cdot \mathbf{p}_1)^3 + \frac{1672}{105} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_1^2 \right) \right. \\ & \left. + \frac{1}{m_1^2} \left( \frac{64}{7} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2) - \frac{1136}{105} (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 - \frac{184}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) \right. \\ & \left. + \frac{1}{m_1 m_2} \left( \frac{76}{7} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2)^2 + \frac{376}{105} (\mathbf{n} \cdot \mathbf{p}_2) (\mathbf{p}_1 \cdot \mathbf{p}_2) - \frac{992}{105} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_2^2 \right) \right. \\ & \left. + \frac{1}{m_2^2} \left( -\frac{208}{35} (\mathbf{n} \cdot \mathbf{p}_2)^3 + \frac{112}{15} (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_2^2 \right) \right] \mathbf{p}_1 + \left[ \frac{1}{m_1^2} \left( \frac{76}{5} (\mathbf{n} \cdot \mathbf{p}_1)^3 - \frac{108}{5} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_1^2 \right) \right. \\ & \left. + \frac{1}{m_1 m_2} \left( -\frac{992}{35} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2) + \frac{664}{35} (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 + \frac{400}{21} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) \right. \\ & \left. + \frac{1}{m_2^2} \left( \frac{292}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2)^2 - \frac{1928}{105} (\mathbf{n} \cdot \mathbf{p}_2) (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{52}{105} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_2^2 \right) \right] \mathbf{p}_2 \Big\} \\ & + \frac{G^3}{r^3} \left\{ \left[ \frac{m_2^2}{m_1} \left( -\frac{7862}{105} (\mathbf{n} \cdot \mathbf{p}_1)^2 + \frac{4129}{105} \mathbf{p}_1^2 \right) + m_2 \left( -\frac{248}{5} (\mathbf{n} \cdot \mathbf{p}_1)^2 + \frac{3756}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) \right. \right. \right. \\ & \left. \left. + \frac{852}{35} \mathbf{p}_1^2 - \frac{2106}{35} (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) + m_1 \left( \frac{5032}{105} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) - \frac{158}{5} (\mathbf{n} \cdot \mathbf{p}_2)^2 - \frac{484}{21} (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{97}{5} \mathbf{p}_2^2 \right) \right] \mathbf{n} \right. \\ & \left. + \left[ \frac{617m_2^2}{15m_1} (\mathbf{n} \cdot \mathbf{p}_1) + m_2 \left( \frac{2588}{105} (\mathbf{n} \cdot \mathbf{p}_1) - \frac{431}{21} (\mathbf{n} \cdot \mathbf{p}_2) \right) - \frac{64}{105} m_1 (\mathbf{n} \cdot \mathbf{p}_2) \right] \mathbf{p}_1 \right. \end{aligned}$$

$$+ \left[ -\frac{1469}{35} m_2 (\mathbf{n} \cdot \mathbf{p}_1) + m_1 \left( -\frac{484}{21} (\mathbf{n} \cdot \mathbf{p}_1) + \frac{103}{5} (\mathbf{n} \cdot \mathbf{p}_2) \right) \right] \mathbf{p}_2 \Big\} + \frac{G^4}{r^4} (m_1^2 m_2^2 + m_1 m_2^3) \mathbf{n} \Big\}, \quad (4.2a)$$

$$\dot{\mathbf{x}}_2 = (1 \rightleftharpoons 2). \quad (4.2b)$$

For the time derivative of the particle momenta, we get

$$\begin{aligned} \dot{\mathbf{p}}_1 = & \frac{1}{c^5} \left\{ \frac{G^3}{r^4} \left\{ \left[ \frac{56}{15} m_1 m_2^2 (\mathbf{n} \cdot \mathbf{p}_1) - \frac{56}{15} m_1^2 m_2 (\mathbf{n} \cdot \mathbf{p}_2) \right] \mathbf{n} - \frac{16}{5} m_1 m_2^2 \mathbf{p}_1 + \frac{16}{5} m_1^2 m_2 \mathbf{p}_2 \right\} \right\} \\ & + \frac{1}{c^7} \left\{ \frac{G^2}{r^3} \left\{ \left[ \frac{m_2}{m_1^3} \left( -10 (\mathbf{n} \cdot \mathbf{p}_1)^5 + \frac{116}{7} (\mathbf{n} \cdot \mathbf{p}_1)^3 \mathbf{p}_1^2 - \frac{198}{35} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_1^4 \right) \right. \right. \right. \\ & + \frac{1}{m_1^2} \left( 30 (\mathbf{n} \cdot \mathbf{p}_1)^4 (\mathbf{n} \cdot \mathbf{p}_2) - \frac{180}{7} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 + \frac{18}{7} (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^4 - \frac{184}{7} (\mathbf{n} \cdot \mathbf{p}_1)^3 (\mathbf{p}_1 \cdot \mathbf{p}_2) \right. \\ & + \frac{568}{35} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) \Big) + \frac{1}{m_1 m_2} \left( -30 (\mathbf{n} \cdot \mathbf{p}_1)^3 (\mathbf{n} \cdot \mathbf{p}_2)^2 + 6 (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2 \right. \\ & + 44 (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2) (\mathbf{p}_1 \cdot \mathbf{p}_2) - \frac{228}{35} (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) - \frac{456}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 \\ & + \frac{30}{7} (\mathbf{n} \cdot \mathbf{p}_1)^3 \mathbf{p}_2^2 - \frac{38}{35} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_1^2 \mathbf{p}_2^2 \Big) + \frac{1}{m_2^2} \left( 10 (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2)^3 + \frac{22}{7} (\mathbf{n} \cdot \mathbf{p}_2)^3 \mathbf{p}_1^2 \right. \\ & - \frac{124}{7} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{208}{35} (\mathbf{n} \cdot \mathbf{p}_2) (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - \frac{30}{7} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_2^2 \\ & \left. \left. \left. - 2 (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 \mathbf{p}_2^2 + \frac{124}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{p}_1 \cdot \mathbf{p}_2) \mathbf{p}_2^2 \right) \right] \mathbf{n} + \left[ \frac{m_2}{m_1^3} \left( -\frac{26}{7} (\mathbf{n} \cdot \mathbf{p}_1)^4 + 4 (\mathbf{n} \cdot \mathbf{p}_1)^2 \mathbf{p}_1^2 - \frac{62}{105} \mathbf{p}_1^4 \right) \right. \right. \\ & + \frac{1}{m_1^2} \left( \frac{40}{7} (\mathbf{n} \cdot \mathbf{p}_1)^3 (\mathbf{n} \cdot \mathbf{p}_2) - \frac{16}{5} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 - \frac{8}{5} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{32}{105} \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) \\ & + \frac{1}{m_1 m_2} \left( -\frac{2}{7} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2)^2 + \frac{2}{7} (\mathbf{n} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2 - \frac{8}{5} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) (\mathbf{p}_1 \cdot \mathbf{p}_2) \right. \\ & + \frac{16}{21} (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + \frac{2}{7} (\mathbf{n} \cdot \mathbf{p}_1)^2 \mathbf{p}_2^2 - \frac{22}{105} \mathbf{p}_1^2 \mathbf{p}_2^2 \Big) + \frac{1}{m_2^2} \left( -\frac{12}{7} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2)^3 + \frac{36}{35} (\mathbf{n} \cdot \mathbf{p}_2)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) \right. \\ & + \frac{4}{5} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_2^2 - \frac{4}{15} (\mathbf{p}_1 \cdot \mathbf{p}_2) \mathbf{p}_2^2 \Big) \Big] \mathbf{p}_1 + \left[ \frac{1}{m_1^2} \left( -\frac{6}{7} (\mathbf{n} \cdot \mathbf{p}_1)^4 + \frac{12}{5} (\mathbf{n} \cdot \mathbf{p}_1)^2 \mathbf{p}_1^2 - \frac{22}{35} \mathbf{p}_1^4 \right) \right. \\ & + \frac{1}{m_1 m_2} \left( \frac{12}{7} (\mathbf{n} \cdot \mathbf{p}_1)^3 (\mathbf{n} \cdot \mathbf{p}_2) - \frac{12}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 - \frac{204}{35} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{28}{15} \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) \\ & + \frac{1}{m_2^2} \left( -\frac{6}{7} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2)^2 - \frac{66}{35} (\mathbf{n} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2 + \frac{192}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) (\mathbf{p}_1 \cdot \mathbf{p}_2) - \frac{40}{21} (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 \right. \\ & + \frac{6}{35} (\mathbf{n} \cdot \mathbf{p}_1)^2 \mathbf{p}_2^2 + \frac{2}{3} \mathbf{p}_1^2 \mathbf{p}_2^2 \Big) \Big] \mathbf{p}_2 \Big\} + \frac{G^3}{r^4} \left\{ \left[ \frac{m_2^2}{m_1} \left( -\frac{52}{5} (\mathbf{n} \cdot \mathbf{p}_1)^3 + \frac{1408}{105} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_1^2 \right) \right. \right. \\ & + m_2 \left( \frac{20}{7} (\mathbf{n} \cdot \mathbf{p}_1)^3 + \frac{84}{5} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2) - \frac{124}{105} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_1^2 - \frac{1388}{105} (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 \right. \\ & \left. \left. - \frac{3128}{105} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) + m_1 \left( -\frac{20}{7} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2) - \frac{464}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2)^2 + \frac{68}{105} (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 \right. \right. \\ & + \frac{24}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{2624}{105} (\mathbf{n} \cdot \mathbf{p}_2) (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{1472}{105} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_2^2 \Big) \\ & \left. \left. + \frac{m_1^2}{m_2} \left( \frac{48}{7} (\mathbf{n} \cdot \mathbf{p}_2)^3 - \frac{324}{35} (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_2^2 \right) \right] \mathbf{n} + \left[ \frac{m_2^2}{m_1} \left( -\frac{62}{21} (\mathbf{n} \cdot \mathbf{p}_1)^2 + \frac{82}{105} \mathbf{p}_1^2 \right) \right. \right. \end{aligned}$$

$$\begin{aligned}
& + m_2 \left( -\frac{128}{105} (\mathbf{n} \cdot \mathbf{p}_1)^2 + \frac{1664}{105} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) + \frac{8}{35} \mathbf{p}_1^2 + \frac{188}{105} (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) \\
& + m_1 \left( \frac{8}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) - \frac{58}{15} (\mathbf{n} \cdot \mathbf{p}_2)^2 + \frac{8}{105} (\mathbf{p}_1 \cdot \mathbf{p}_2) - \frac{34}{105} \mathbf{p}_2^2 \right) \mathbf{p}_1 \\
& + \left[ m_2 \left( \frac{134}{21} (\mathbf{n} \cdot \mathbf{p}_1)^2 - \frac{254}{105} \mathbf{p}_1^2 \right) + m_1 \left( \frac{148}{105} (\mathbf{n} \cdot \mathbf{p}_1)^2 - \frac{2024}{105} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) \right. \right. \\
& \left. \left. - \frac{52}{105} \mathbf{p}_1^2 + \frac{292}{105} (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) + \frac{m_1^2}{m_2} \left( \frac{30}{7} (\mathbf{n} \cdot \mathbf{p}_2)^2 - \frac{14}{5} \mathbf{p}_2^2 \right) \right] \mathbf{p}_2 \Big\} + \frac{G^4}{r^5} \left\{ \left[ -\frac{3574}{105} m_1 m_2^3 (\mathbf{n} \cdot \mathbf{p}_1) \right. \right. \\
& \left. \left. + \frac{174}{5} m_1^3 m_2 (\mathbf{n} \cdot \mathbf{p}_2) + m_1^2 m_2^2 \left( -\frac{3838}{105} (\mathbf{n} \cdot \mathbf{p}_1) + \frac{3782}{105} (\mathbf{n} \cdot \mathbf{p}_2) \right) \right] \mathbf{n} \right. \\
& \left. + \left[ \frac{692}{21} m_1^2 m_2^2 + \frac{3268}{105} m_1 m_2^3 \right] \mathbf{p}_1 + \left[ -\frac{1116}{35} m_1^3 m_2 - \frac{164}{5} m_1^2 m_2^2 \right] \mathbf{p}_2 \right\} \Big\}, \tag{4.3a}
\end{aligned}$$

$$\dot{\mathbf{p}}_2 = (1 \Rightarrow 2). \tag{4.3b}$$

To check the correctness of the expressions for  $\dot{\mathbf{x}}_a$ ,  $\dot{\mathbf{p}}_a$ , we employ them to compute the instantaneous energy loss of a binary, at 2.5PN and 3.5PN orders by differentiating 1PN accurate conservative Hamiltonian expression [see, Hamiltonian (2.5) of Ref. [15] truncated at the 1PN level].<sup>4</sup> The contribution to the Hamiltonian above comes purely from the matter part and it reads

$$\begin{aligned}
H_{\leq 1\text{PN}}^{\text{mat}}(\mathbf{x}_a, \mathbf{p}_a) = & \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} - \frac{Gm_1 m_2}{r} - \frac{1}{8c^2} \left( \frac{\mathbf{p}_1^4}{m_1^3} + \frac{\mathbf{p}_2^4}{m_2^3} \right) + \frac{Gm_1 m_2}{2rc^2} \left[ -3 \left( \frac{\mathbf{p}_1^2}{m_1^2} + \frac{\mathbf{p}_2^2}{m_2^2} \right) \right. \\
& \left. + 7 \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + \frac{(\mathbf{n} \cdot \mathbf{p}_1)(\mathbf{n} \cdot \mathbf{p}_2)}{m_1 m_2} \right] + \frac{G^2 m_1 m_2 (m_1 + m_2)}{2r^2 c^2}. \tag{4.4}
\end{aligned}$$

Differentiation with respect to time leads to

$$\frac{d}{dt} H_{\leq 1\text{PN}}^{\text{mat}} = \frac{\partial H_{\leq 1\text{PN}}^{\text{mat}}}{\partial \mathbf{x}_1} \dot{\mathbf{x}}_1 + \frac{\partial H_{\leq 1\text{PN}}^{\text{mat}}}{\partial \mathbf{x}_2} \dot{\mathbf{x}}_2 + \frac{\partial H_{\leq 1\text{PN}}^{\text{mat}}}{\partial \mathbf{p}_1} \dot{\mathbf{p}}_1 + \frac{\partial H_{\leq 1\text{PN}}^{\text{mat}}}{\partial \mathbf{p}_2} \dot{\mathbf{p}}_2. \tag{4.5}$$

We then replace the time derivatives of  $\mathbf{x}_1$ ,  $\mathbf{x}_2$ ,  $\mathbf{p}_1$  and  $\mathbf{p}_2$  occurring in Eq. (4.5) by their expressions (4.2) and (4.3). At last, we write the particle momenta in terms of the particle coordinate velocities with the help of relations (3.8). We introduce the relative velocity, the relative position, the total mass and the parameter  $\nu$ , and multiply by  $-1$  according to definition (3.5). In this way, we obtain for the luminosity [24]:

$$\begin{aligned}
\mathcal{L}_{\leq 3.5\text{PN}}^{(2)} = & -\frac{d}{dt} H_{\leq 1\text{PN}}^{\text{mat}} = \frac{4}{15} \frac{G^2 M^3 \nu^2}{c^5 r^3} \left\{ \left[ 24 \mathbf{v}^2 - 22 (\mathbf{n} \cdot \mathbf{v})^2 \right] \frac{GM}{r} \right\} \\
& + \frac{1}{105} \frac{G^2 M^3 \nu^2}{c^7 r^3} \left\{ -105 \frac{G^3 M^3}{r^3} + \left[ -(4709 + 384\nu) \mathbf{v}^2 + (4821 + 280\nu) (\mathbf{n} \cdot \mathbf{v})^2 \right] \frac{G^2 M^2}{r^2} \right. \\
& + \left[ (914 - 784\nu) \mathbf{v}^4 - 2(1313 - 1894\nu) (\mathbf{n} \cdot \mathbf{v})^2 \mathbf{v}^2 + 3(508 - 1012\nu) (\mathbf{n} \cdot \mathbf{v})^4 \right] \frac{GM}{r} \\
& + \left[ (62 - 344\nu) \mathbf{v}^6 + 3(58 + 284\nu) (\mathbf{n} \cdot \mathbf{v})^2 \mathbf{v}^4 - 450(3 - 4\nu) (\mathbf{n} \cdot \mathbf{v})^4 \mathbf{v}^2 \right. \\
& \left. \left. + 1050(1 - 2\nu) (\mathbf{n} \cdot \mathbf{v})^6 \right] \right\}. \tag{4.6}
\end{aligned}$$

Though this result differs from the instantaneous energy loss (3.14), one will see by using the method presented

<sup>4</sup> The 2PN terms do not play any role in the computation of the 2.5PN and 3.5PN energy loss, since they first couple with the reaction force at the  $(2 + 2.5)\text{PN} = 4.5\text{PN}$  order.

in Ref. [26] for averaging that it leads to the same averaged value, in agreement with Eqs. (4.20) and (4.21) of

Ref. [26].

## V. EULER-LAGRANGIAN EQUATIONS OF MOTIONS

The aim of the present section is to compute the PN reactive corrections to  $\ddot{\mathbf{x}}_a$  employing two slightly different methods. We will see that the computation of these Euler-Lagrangian equations of motion does not involve a Lagrangian in either of these methods. However, we will require PN accurate relations between particle momenta and its coordinate velocities. We turn to derive these relations in the next subsection.

### A. Legendre transformation

For the present computation, we require 1PN, 2.5PN and 3.5PN corrections to the familiar  $\mathbf{p}_a = m_a \mathbf{v}_a$  relations. So, we have to invert the equations (4.2) giving  $\dot{\mathbf{x}}_1 = f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2)$  and  $\dot{\mathbf{x}}_2$  respectively, in order to determine  $\mathbf{p}_1 = f(\mathbf{x}_1, \mathbf{x}_2, \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2) = f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2)$  and  $\mathbf{p}_2$ . This will provide the link between the particle mo-

menta and the particle coordinate velocities up to the order  $1/c^7$ .

As the first step, we add Newtonian and 1PN contributions to  $\dot{\mathbf{x}}_a$  to the right-hand side of Eqs. (4.2). These corrections are obtained from Eqs. (3.7a) and (3.7b) after replacing primed quantities by their unprimed counterparts. Through the expression presented below refer to the first body, it should be noted that similar expressions hold true for the second body. It is the full coupled system that must be inverted. The time derivative of the position  $\mathbf{x}_1$  has the form

$$\dot{\mathbf{x}}_1 = \mathbf{v}_1 = \frac{\mathbf{p}_1}{m_1} + \frac{1}{c^2} A_2 + \frac{1}{c^5} A_5 + \frac{1}{c^7} A_7, \quad (5.1)$$

whereas the structure of  $\mathbf{p}_1$  after inversion is

$$\mathbf{p}_1 = m_1 \mathbf{v}_1 + \frac{1}{c^2} B_2 + \frac{1}{c^5} B_5 + \frac{1}{c^7} B_7. \quad (5.2)$$

The coefficients  $B_2, B_5$  and  $B_7$  in Eq. (5.2) are computed iteratively in terms of  $A_2, A_5$  and  $A_7$  of by means of Eq. (5.1). The first-order coefficient  $B_2$  is read directly from Eq. (3.8a). We get [24]:

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$$\begin{aligned} \mathbf{p}_1 = m_1 \mathbf{v}_1 &+ \frac{1}{c^2} \left\{ \frac{1}{2} m_1 \mathbf{v}_1^2 \mathbf{v}_1 + \frac{G m_1 m_2}{2r} \left\{ -(\mathbf{n} \cdot \mathbf{v}_2) \mathbf{n} + 6 \mathbf{v}_1 - 7 \mathbf{v}_2 \right\} \right\} + \frac{1}{c^5} \left\{ \frac{G^2 m_1^2 m_2}{r^2} \left\{ \left[ -\frac{24}{5} (\mathbf{n} \cdot \mathbf{v}_1)^2 \right. \right. \right. \\ &+ \frac{24}{5} (\mathbf{n} \cdot \mathbf{v}_1) (\mathbf{n} \cdot \mathbf{v}_2) + \frac{16}{5} \mathbf{v}_1^2 - \frac{16}{5} (\mathbf{v}_1 \cdot \mathbf{v}_2) \Big] \mathbf{n} + \left[ \frac{8}{3} (\mathbf{n} \cdot \mathbf{v}_1) + \frac{8}{15} (\mathbf{n} \cdot \mathbf{v}_2) \right] \mathbf{v}_1 - \frac{16}{5} (\mathbf{n} \cdot \mathbf{v}_1) \mathbf{v}_2 \Big\} \Big\} \\ &+ \frac{1}{c^7} \left\{ \frac{G^2 m_1^2 m_2}{r^2} \left\{ \left[ -\frac{36}{7} (\mathbf{n} \cdot \mathbf{v}_1)^4 + \frac{148}{7} (\mathbf{n} \cdot \mathbf{v}_1)^3 (\mathbf{n} \cdot \mathbf{v}_2) - \frac{104}{7} (\mathbf{n} \cdot \mathbf{v}_1)^2 (\mathbf{n} \cdot \mathbf{v}_2)^2 - \frac{8}{7} (\mathbf{n} \cdot \mathbf{v}_1) (\mathbf{n} \cdot \mathbf{v}_2)^3 \right. \right. \right. \\ &+ \frac{164}{35} \mathbf{v}_1^4 + \left( -\frac{12}{7} (\mathbf{n} \cdot \mathbf{v}_1)^2 + \frac{432}{35} (\mathbf{n} \cdot \mathbf{v}_1) (\mathbf{n} \cdot \mathbf{v}_2) - \frac{292}{35} (\mathbf{n} \cdot \mathbf{v}_2)^2 \right) (\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{136}{35} (\mathbf{v}_1 \cdot \mathbf{v}_2)^2 \\ &+ \left( -\frac{132}{35} (\mathbf{n} \cdot \mathbf{v}_1)^2 + \frac{52}{7} (\mathbf{n} \cdot \mathbf{v}_1) (\mathbf{n} \cdot \mathbf{v}_2) - \frac{44}{21} (\mathbf{v}_1 \cdot \mathbf{v}_2) \right) \mathbf{v}_2^2 + \left( \frac{48}{35} (\mathbf{n} \cdot \mathbf{v}_1)^2 - \frac{116}{7} (\mathbf{n} \cdot \mathbf{v}_1) (\mathbf{n} \cdot \mathbf{v}_2) \right. \\ &+ \frac{324}{35} (\mathbf{n} \cdot \mathbf{v}_2)^2 - \frac{284}{35} (\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{172}{105} \mathbf{v}_2^2 \Big] \mathbf{v}_1^2 \Big\} + \left[ \frac{156}{35} (\mathbf{n} \cdot \mathbf{v}_1)^3 - \frac{152}{35} (\mathbf{n} \cdot \mathbf{v}_1)^2 (\mathbf{n} \cdot \mathbf{v}_2) \right. \\ &- \frac{76}{7} (\mathbf{n} \cdot \mathbf{v}_1) (\mathbf{n} \cdot \mathbf{v}_2)^2 + \frac{208}{35} (\mathbf{n} \cdot \mathbf{v}_2)^3 + \left( -\frac{212}{35} (\mathbf{n} \cdot \mathbf{v}_1) + \frac{416}{35} (\mathbf{n} \cdot \mathbf{v}_2) \right) \mathbf{v}_1^2 \\ &+ \left( -\frac{8}{7} (\mathbf{n} \cdot \mathbf{v}_1) - \frac{376}{105} (\mathbf{n} \cdot \mathbf{v}_2) \right) (\mathbf{v}_1 \cdot \mathbf{v}_2) + \left( \frac{992}{105} (\mathbf{n} \cdot \mathbf{v}_1) - \frac{36}{5} (\mathbf{n} \cdot \mathbf{v}_2) \right) \mathbf{v}_2^2 \Big] \mathbf{v}_1 \\ &+ \left[ -\frac{76}{5} (\mathbf{n} \cdot \mathbf{v}_1)^3 + \frac{992}{35} (\mathbf{n} \cdot \mathbf{v}_1)^2 (\mathbf{n} \cdot \mathbf{v}_2) - \frac{292}{35} (\mathbf{n} \cdot \mathbf{v}_1) (\mathbf{n} \cdot \mathbf{v}_2)^2 + \left( \frac{92}{5} (\mathbf{n} \cdot \mathbf{v}_1) - \frac{664}{35} (\mathbf{n} \cdot \mathbf{v}_2) \right) \mathbf{v}_1^2 \right. \\ &+ \left( -\frac{400}{21} (\mathbf{n} \cdot \mathbf{v}_1) + \frac{1928}{105} (\mathbf{n} \cdot \mathbf{v}_2) \right) (\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{44}{21} (\mathbf{n} \cdot \mathbf{v}_1) \mathbf{v}_2^2 \Big] \mathbf{v}_2 \Big\} \\ &+ \frac{G^3}{r^3} \left\{ \left[ m_1^3 m_2 \left( \frac{168}{5} (\mathbf{n} \cdot \mathbf{v}_1)^2 - \frac{704}{21} (\mathbf{n} \cdot \mathbf{v}_1) (\mathbf{n} \cdot \mathbf{v}_2) - \frac{92}{7} \mathbf{v}_1^2 + \frac{1412}{105} (\mathbf{v}_1 \cdot \mathbf{v}_2) \right) \right. \right. \end{aligned}$$

$$\begin{aligned}
& + m_1^2 m_2^2 \left( \frac{3326}{105} (\mathbf{n} \cdot \mathbf{v}_1)^2 - \frac{2812}{105} (\mathbf{n} \cdot \mathbf{v}_1) (\mathbf{n} \cdot \mathbf{v}_2) - \frac{62}{15} (\mathbf{n} \cdot \mathbf{v}_2)^2 - \frac{221}{21} \mathbf{v}_1^2 + \frac{202}{35} (\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{23}{5} \mathbf{v}_2^2 \right) \mathbf{n} \\
& + \left[ m_1^2 m_2^2 \left( -\frac{257}{15} (\mathbf{n} \cdot \mathbf{v}_1) + \frac{13}{7} (\mathbf{n} \cdot \mathbf{v}_2) \right) + m_1^3 m_2 \left( -\frac{1636}{105} (\mathbf{n} \cdot \mathbf{v}_1) + \frac{232}{105} (\mathbf{n} \cdot \mathbf{v}_2) \right) \right] \mathbf{v}_1 \\
& + \left[ \frac{1412}{105} m_1^3 m_2 (\mathbf{n} \cdot \mathbf{v}_1) + m_1^2 m_2^2 \left( \frac{1607}{105} (\mathbf{n} \cdot \mathbf{v}_1) - \frac{1}{3} (\mathbf{n} \cdot \mathbf{v}_2) \right) \right] \mathbf{v}_2 \Big\} + \frac{G^4}{r^4} \left\{ \left[ -m_1^3 m_2^2 - m_1^2 m_2^3 \right] \mathbf{n} \right\}, \quad (5.3a)
\end{aligned}$$

$$\mathbf{p}_2 = (1 \rightleftharpoons 2). \quad (5.3b)$$

### B. First method to calculate the Euler-Lagrangian equations of motions

The calculation of the accelerations makes the connection with the Euler-Lagrangian approach. It is achieved in a first stage by evaluating the total time derivatives of the functions  $\dot{\mathbf{x}}_1 = f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2)$  and  $\dot{\mathbf{x}}_2 = f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2)$  as given by the Eqs. (4.2). It is worth stressing that this method does not need the explicit knowledge of the Lagrangian. We can write indeed:

$$\ddot{\mathbf{x}}_1 = \frac{d}{dt} \dot{\mathbf{x}}_1 = \frac{\partial \dot{\mathbf{x}}_1}{\partial \mathbf{x}_1} \dot{\mathbf{x}}_1 + \frac{\partial \dot{\mathbf{x}}_1}{\partial \mathbf{x}_2} \dot{\mathbf{x}}_2 + \frac{\partial \dot{\mathbf{x}}_1}{\partial \mathbf{p}_1} \dot{\mathbf{p}}_1 + \frac{\partial \dot{\mathbf{x}}_1}{\partial \mathbf{p}_2} \dot{\mathbf{p}}_2, \quad (5.4a)$$

$$\ddot{\mathbf{x}}_2 = \frac{d}{dt} \dot{\mathbf{x}}_2 = \frac{\partial \dot{\mathbf{x}}_2}{\partial \mathbf{x}_1} \dot{\mathbf{x}}_1 + \frac{\partial \dot{\mathbf{x}}_2}{\partial \mathbf{x}_2} \dot{\mathbf{x}}_2 + \frac{\partial \dot{\mathbf{x}}_2}{\partial \mathbf{p}_1} \dot{\mathbf{p}}_1 + \frac{\partial \dot{\mathbf{x}}_2}{\partial \mathbf{p}_2} \dot{\mathbf{p}}_2. \quad (5.4b)$$

Inserting appropriate PN contributions to  $\dot{\mathbf{x}}_a$  and  $\dot{\mathbf{p}}_a$  given in Eqs. (4.2) and (4.3), we obtain 1PN, 2.5PN and 3.5PN corrections to the familiar Newtonian expression for  $\ddot{\mathbf{x}}_a$  and it reads — apart from the 2PN and 3PN terms — [24]:

$$\begin{aligned}
\ddot{\mathbf{x}}_1 = & -\frac{G m_2}{r^2} \mathbf{n} + \frac{1}{c^2} \left\{ \frac{G}{r^2} \left\{ \left[ -\frac{m_2}{m_1^2} \mathbf{p}_1^2 + \frac{4}{m_1} (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{1}{m_2} \left( \frac{3}{2} (\mathbf{n} \cdot \mathbf{p}_2)^2 - 2 \mathbf{p}_2^2 \right) \right] \mathbf{n} + \left[ 4 \frac{m_2}{m_1^2} (\mathbf{n} \cdot \mathbf{p}_1) \right. \right. \right. \\
& \left. \left. - 3 \frac{1}{m_1} (\mathbf{n} \cdot \mathbf{p}_2) \right] \mathbf{p}_1 + \left[ -4 \frac{1}{m_1} (\mathbf{n} \cdot \mathbf{p}_1) + 3 \frac{1}{m_2} (\mathbf{n} \cdot \mathbf{p}_2) \right] \mathbf{p}_2 \right\} + \frac{G^2}{r^3} \left[ 5 m_1 m_2 + 4 m_2^2 \right] \mathbf{n} \Big\} \\
& + \frac{1}{c^5} \left\{ \frac{G^2}{r^3} \left\{ \left[ \frac{m_2}{m_1^2} \left( -24 (\mathbf{n} \cdot \mathbf{p}_1)^3 + \frac{96}{5} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_1^2 \right) + \frac{1}{m_1} \left( 48 (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2) - \frac{72}{5} (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 \right. \right. \right. \right. \\
& \left. \left. - 24 (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) + \frac{1}{m_2} \left( \frac{72}{5} (\mathbf{n} \cdot \mathbf{p}_2) (\mathbf{p}_1 \cdot \mathbf{p}_2) + (\mathbf{n} \cdot \mathbf{p}_1) \left( -24 (\mathbf{n} \cdot \mathbf{p}_2)^2 + \frac{24}{5} \mathbf{p}_2^2 \right) \right) \right] \mathbf{n} \right. \\
& \left. + \left[ \frac{m_2}{m_1^2} \left( \frac{64}{5} (\mathbf{n} \cdot \mathbf{p}_1)^2 - \frac{88}{15} \mathbf{p}_1^2 \right) + \frac{1}{m_1} \left( -\frac{56}{5} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) + \frac{16}{3} (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) + \frac{1}{m_2} \left( -\frac{8}{5} (\mathbf{n} \cdot \mathbf{p}_2)^2 \right. \right. \right. \\
& \left. \left. + \frac{8}{15} \mathbf{p}_2^2 \right) \right] \mathbf{p}_1 + \left[ \frac{1}{m_1} \left( -\frac{72}{5} (\mathbf{n} \cdot \mathbf{p}_1)^2 + \frac{32}{5} \mathbf{p}_1^2 \right) + \frac{1}{m_2} \left( \frac{72}{5} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) - \frac{32}{5} (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) \right] \mathbf{p}_2 \Big\} \\
& + \frac{G^3}{r^4} \left\{ \left[ \frac{16}{5} m_2^2 (\mathbf{n} \cdot \mathbf{p}_1) + m_1 m_2 \left( \frac{8}{5} (\mathbf{n} \cdot \mathbf{p}_1) - \frac{8}{5} (\mathbf{n} \cdot \mathbf{p}_2) \right) \right] \mathbf{n} - \left[ \frac{8}{15} m_1 m_2 + \frac{8}{15} m_2^2 \right] \mathbf{p}_1 \right\} \\
& + \frac{1}{c^7} \left\{ \frac{G^2}{r^3} \left\{ \left[ \frac{m_2}{m_1^4} \left( -46 (\mathbf{n} \cdot \mathbf{p}_1)^5 + 92 (\mathbf{n} \cdot \mathbf{p}_1)^3 \mathbf{p}_1^2 - \frac{1146}{35} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_1^4 \right) + \frac{1}{m_1^3} \left( 214 (\mathbf{n} \cdot \mathbf{p}_1)^4 (\mathbf{n} \cdot \mathbf{p}_2) \right. \right. \right. \right. \\
& \left. \left. - \frac{1756}{7} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 + \frac{934}{35} (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^4 - \frac{536}{7} (\mathbf{n} \cdot \mathbf{p}_1)^3 (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{1772}{35} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) \right. \\
& \left. + \frac{1}{m_1^2 m_2} \left( \frac{1452}{7} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2) (\mathbf{p}_1 \cdot \mathbf{p}_2) - 44 (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) + \left( -282 (\mathbf{n} \cdot \mathbf{p}_2)^2 + \frac{46}{7} \mathbf{p}_2^2 \right) (\mathbf{n} \cdot \mathbf{p}_1)^3 \right. \right. \\
& \left. \left. + \left( \frac{1322}{7} (\mathbf{n} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2 - \frac{120}{7} (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - 10 \mathbf{p}_1^2 \mathbf{p}_2^2 \right) (\mathbf{n} \cdot \mathbf{p}_1) \right) + \frac{1}{m_1 m_2^2} \left( -42 (\mathbf{n} \cdot \mathbf{p}_2)^3 \mathbf{p}_1^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + (\mathbf{n} \cdot \mathbf{p}_1)^2 \left( 106 (\mathbf{n} \cdot \mathbf{p}_2)^3 - 2 (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_2^2 \right) + \left( \frac{816}{35} (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 + \frac{398}{35} \mathbf{p}_1^2 \mathbf{p}_2^2 \right) (\mathbf{n} \cdot \mathbf{p}_2) \\
& + \left( -\frac{1032}{7} (\mathbf{n} \cdot \mathbf{p}_2)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{232}{35} (\mathbf{p}_1 \cdot \mathbf{p}_2) \mathbf{p}_2^2 \right) (\mathbf{n} \cdot \mathbf{p}_1) + \frac{1}{m_2^3} \left( \frac{284}{7} (\mathbf{n} \cdot \mathbf{p}_2)^3 (\mathbf{p}_1 \cdot \mathbf{p}_2) \right. \\
& - \frac{608}{35} (\mathbf{n} \cdot \mathbf{p}_2) (\mathbf{p}_1 \cdot \mathbf{p}_2) \mathbf{p}_2^2 + \left. \left( 8 (\mathbf{n} \cdot \mathbf{p}_2)^4 - \frac{116}{7} (\mathbf{n} \cdot \mathbf{p}_2)^2 \mathbf{p}_2^2 + \frac{92}{35} \mathbf{p}_2^4 \right) (\mathbf{n} \cdot \mathbf{p}_1) \right) \mathbf{n} \\
& + \left[ \frac{m_2}{m_1^4} \left( \frac{334}{7} (\mathbf{n} \cdot \mathbf{p}_1)^4 - \frac{3028}{35} (\mathbf{n} \cdot \mathbf{p}_1)^2 \mathbf{p}_1^2 + \frac{386}{21} \mathbf{p}_1^4 \right) + \frac{1}{m_1^3} \left( -\frac{752}{7} (\mathbf{n} \cdot \mathbf{p}_1)^3 (\mathbf{n} \cdot \mathbf{p}_2) \right. \right. \\
& + \frac{828}{7} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 + \frac{264}{5} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) - \frac{984}{35} \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) \left. \right) + \frac{1}{m_2^3} \left( -\frac{208}{7} (\mathbf{n} \cdot \mathbf{p}_2)^4 \right. \\
& + \frac{1436}{35} (\mathbf{n} \cdot \mathbf{p}_2)^2 \mathbf{p}_2^2 - \frac{116}{15} \mathbf{p}_2^4 \left. \right) + \frac{1}{m_1^2 m_2} \left( -\frac{214}{7} (\mathbf{n} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2 - \frac{1296}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) (\mathbf{p}_1 \cdot \mathbf{p}_2) \right. \\
& + \frac{40}{7} (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - \frac{10}{21} \mathbf{p}_1^2 \mathbf{p}_2^2 + \left. \left( 6 (\mathbf{n} \cdot \mathbf{p}_2)^2 + \frac{814}{35} \mathbf{p}_2^2 \right) (\mathbf{n} \cdot \mathbf{p}_1)^2 \right) + \frac{1}{m_1 m_2^2} \left( -\frac{60}{7} (\mathbf{n} \cdot \mathbf{p}_2)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) \right. \\
& + \frac{428}{35} (\mathbf{p}_1 \cdot \mathbf{p}_2) \mathbf{p}_2^2 + \left. \left( \frac{584}{7} (\mathbf{n} \cdot \mathbf{p}_2)^3 - \frac{2544}{35} (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_2^2 \right) (\mathbf{n} \cdot \mathbf{p}_1) \right) \mathbf{p}_1 + \left[ \frac{1}{m_1^3} \left( -82 (\mathbf{n} \cdot \mathbf{p}_1)^4 \right. \right. \\
& + \frac{4416}{35} (\mathbf{n} \cdot \mathbf{p}_1)^2 \mathbf{p}_1^2 - \frac{838}{35} \mathbf{p}_1^4 \left. \right) + \frac{1}{m_1^2 m_2} \left( \frac{1684}{7} (\mathbf{n} \cdot \mathbf{p}_1)^3 (\mathbf{n} \cdot \mathbf{p}_2) - \frac{7004}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 \right. \\
& - \frac{4852}{35} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{396}{7} \mathbf{p}_1^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) \left. \right) + \frac{1}{m_1 m_2^2} \left( \frac{2542}{35} (\mathbf{n} \cdot \mathbf{p}_2)^2 \mathbf{p}_1^2 \right. \\
& + \frac{1424}{7} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) (\mathbf{p}_1 \cdot \mathbf{p}_2) - \frac{248}{7} (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - \frac{622}{35} \mathbf{p}_1^2 \mathbf{p}_2^2 + \left. \left( -\frac{1394}{7} (\mathbf{n} \cdot \mathbf{p}_2)^2 + \frac{898}{35} \mathbf{p}_2^2 \right) (\mathbf{n} \cdot \mathbf{p}_1)^2 \right) \\
& + \frac{1}{m_2^3} \left( -\frac{2512}{35} (\mathbf{n} \cdot \mathbf{p}_2)^2 (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{144}{7} (\mathbf{p}_1 \cdot \mathbf{p}_2) \mathbf{p}_2^2 + \left. \left( \frac{284}{7} (\mathbf{n} \cdot \mathbf{p}_2)^3 - \frac{608}{35} (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_2^2 \right) (\mathbf{n} \cdot \mathbf{p}_1) \right) \right] \mathbf{p}_2 \Big\} \\
& + \frac{G^3}{r^4} \left\{ \left[ -\frac{5316}{7} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2) + \frac{m_2^2}{m_1^2} \left( \frac{18084}{35} (\mathbf{n} \cdot \mathbf{p}_1)^3 - \frac{2472}{7} (\mathbf{n} \cdot \mathbf{p}_1) \mathbf{p}_1^2 \right) + \left( \frac{19984}{105} \mathbf{p}_1^2 \right. \right. \right. \\
& - \frac{45592}{105} (\mathbf{p}_1 \cdot \mathbf{p}_2) \left. \right) (\mathbf{n} \cdot \mathbf{p}_2) + \frac{m_2}{m_1} \left( \frac{13476}{35} (\mathbf{n} \cdot \mathbf{p}_1)^3 - \frac{43208}{35} (\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2) + \frac{31184}{105} (\mathbf{n} \cdot \mathbf{p}_2) \mathbf{p}_1^2 \right. \\
& + \left. \left( -\frac{3916}{15} \mathbf{p}_1^2 + \frac{56752}{105} (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) (\mathbf{n} \cdot \mathbf{p}_1) \right) + \left( \frac{6284}{7} (\mathbf{n} \cdot \mathbf{p}_2)^2 + \frac{952}{3} (\mathbf{p}_1 \cdot \mathbf{p}_2) - \frac{19276}{105} \mathbf{p}_2^2 \right) (\mathbf{n} \cdot \mathbf{p}_1) \\
& + \frac{m_1}{m_2} \left( -\frac{6464}{35} (\mathbf{n} \cdot \mathbf{p}_2)^3 + \left( \frac{1848}{5} (\mathbf{n} \cdot \mathbf{p}_2)^2 - \frac{6716}{105} \mathbf{p}_2^2 \right) (\mathbf{n} \cdot \mathbf{p}_1) + \left( -184 (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{13868}{105} \mathbf{p}_2^2 \right) (\mathbf{n} \cdot \mathbf{p}_2) \right) \Big] \mathbf{n} \\
& + \left[ \frac{6724}{35} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) - \frac{1768}{15} (\mathbf{n} \cdot \mathbf{p}_2)^2 - \frac{2104}{35} (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{4366}{105} \mathbf{p}_2^2 + \frac{m_2^2}{m_1^2} \left( -\frac{9876}{35} (\mathbf{n} \cdot \mathbf{p}_1)^2 + \frac{2874}{35} \mathbf{p}_1^2 \right) \right. \\
& + \frac{m_2}{m_1} \left( -\frac{20264}{105} (\mathbf{n} \cdot \mathbf{p}_1)^2 + \frac{6124}{15} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) + \frac{872}{15} \mathbf{p}_1^2 - \frac{13172}{105} (\mathbf{p}_1 \cdot \mathbf{p}_2) \right) + \frac{m_1}{m_2} \left( -\frac{8}{15} (\mathbf{n} \cdot \mathbf{p}_2)^2 \right. \\
& + \frac{596}{105} \mathbf{p}_2^2 \left. \right) \Big] \mathbf{p}_1 + \left[ \frac{19604}{105} (\mathbf{n} \cdot \mathbf{p}_1)^2 - \frac{43328}{105} (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) - \frac{348}{7} \mathbf{p}_1^2 + \frac{12356}{105} (\mathbf{p}_1 \cdot \mathbf{p}_2) + \frac{m_2}{m_1} \left( \frac{30052}{105} (\mathbf{n} \cdot \mathbf{p}_1)^2 \right. \right. \\
& - \frac{2734}{35} \mathbf{p}_1^2 \left. \right) + \frac{m_1}{m_2} \left( -184 (\mathbf{n} \cdot \mathbf{p}_1) (\mathbf{n} \cdot \mathbf{p}_2) + \frac{12548}{105} (\mathbf{n} \cdot \mathbf{p}_2)^2 + \frac{704}{15} (\mathbf{p}_1 \cdot \mathbf{p}_2) - \frac{3874}{105} \mathbf{p}_2^2 \right) \Big] \mathbf{p}_2 \Big\} \\
& + \frac{G^4}{r^5} \left\{ \left[ -\frac{296}{15} m_2^3 (\mathbf{n} \cdot \mathbf{p}_1) + m_1^2 m_2 \left( \frac{8}{35} (\mathbf{n} \cdot \mathbf{p}_1) + \frac{344}{15} (\mathbf{n} \cdot \mathbf{p}_2) \right) + m_1 m_2^2 \left( -\frac{872}{35} (\mathbf{n} \cdot \mathbf{p}_1) + \frac{352}{15} (\mathbf{n} \cdot \mathbf{p}_2) \right) \right] \mathbf{n} \right.
\end{aligned}$$

$$+ \left[ -\frac{8}{105}m_1^2 m_2 - \frac{226}{105}m_1 m_2^2 - \frac{218}{105}m_2^3 \right] \mathbf{p}_1 + \left[ \frac{58}{21}m_1^2 m_2 + \frac{62}{35}m_1 m_2^2 \right] \mathbf{p}_2 \Big\} \Bigg\}, \quad (5.5a)$$

$$\ddot{\mathbf{x}}_2 = (1 \Rightarrow 2). \quad (5.5b)$$

We now go on to present  $\ddot{\mathbf{x}}_a$  in terms of  $(\mathbf{x}_a, \mathbf{v}_a)$ . As Eqs. (5.5) involve  $\mathbf{p}_a$  at 1PN, 2.5PN and 3.5PN orders, we use only Newtonian, 1PN, 2.5PN contributions to  $\mathbf{p}_a$ , given in Eqs. (5.3) [rather than Eqs. (3.8) truncated at the 1PN approximation]. After some heavy algebra, we get  $\ddot{\mathbf{x}}_1 = f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2)$  and  $\ddot{\mathbf{x}}_2 = f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2)$  [24]:

$$\begin{aligned} \ddot{\mathbf{x}}_1 = & -\frac{Gm_2}{r^2}\mathbf{n} + \frac{1}{c^2}\left\{ \frac{Gm_2}{r^2}\left\{ \left[ \frac{3}{2}(\mathbf{n} \cdot \mathbf{v}_2)^2 - \mathbf{v}_1^2 + 4(\mathbf{v}_1 \cdot \mathbf{v}_2) - 2\mathbf{v}_2^2 \right] \mathbf{n} + \left[ 4(\mathbf{n} \cdot \mathbf{v}_1) - 3(\mathbf{n} \cdot \mathbf{v}_2) \right] (\mathbf{v}_1 - \mathbf{v}_2) \right\} \right. \\ & + \frac{G^2 m_2}{r^3} \left[ 5m_1 + 4m_2 \right] \mathbf{n} \Big\} + \frac{1}{c^5}\left\{ \frac{G^2 m_1 m_2}{r^3} \left\{ \left[ -24(\mathbf{n} \cdot \mathbf{v}_1)^3 + \left( 48(\mathbf{n} \cdot \mathbf{v}_1)^2 - \frac{72}{5}\mathbf{v}_1^2 + \frac{72}{5}(\mathbf{v}_1 \cdot \mathbf{v}_2) \right) (\mathbf{n} \cdot \mathbf{v}_2) \right. \right. \right. \\ & + \left( -24(\mathbf{n} \cdot \mathbf{v}_2)^2 + \frac{96}{5}\mathbf{v}_1^2 - 24(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{24}{5}\mathbf{v}_2^2 \right) (\mathbf{n} \cdot \mathbf{v}_1) \Big] \mathbf{n} + \left[ \frac{64}{5}(\mathbf{n} \cdot \mathbf{v}_1)^2 - \frac{56}{5}(\mathbf{n} \cdot \mathbf{v}_1)(\mathbf{n} \cdot \mathbf{v}_2) \right. \\ & - \frac{8}{5}(\mathbf{n} \cdot \mathbf{v}_2)^2 - \frac{88}{15}\mathbf{v}_1^2 + \frac{16}{3}(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{8}{15}\mathbf{v}_2^2 \Big] \mathbf{v}_1 + \left[ -\frac{72}{5}(\mathbf{n} \cdot \mathbf{v}_1)^2 + \frac{72}{5}(\mathbf{n} \cdot \mathbf{v}_1)(\mathbf{n} \cdot \mathbf{v}_2) + \frac{32}{5}\mathbf{v}_1^2 \right. \\ & \left. \left. \left. - \frac{32}{5}(\mathbf{v}_1 \cdot \mathbf{v}_2) \right] \mathbf{v}_2 \right\} + \frac{G^3 m_1 m_2}{r^4} \left\{ \left[ \frac{8}{5}m_1(\mathbf{n} \cdot \mathbf{v}_1) + m_2 \left( \frac{16}{5}(\mathbf{n} \cdot \mathbf{v}_1) - \frac{8}{5}(\mathbf{n} \cdot \mathbf{v}_2) \right) \right] \mathbf{n} - \left[ \frac{8}{15}m_1 + \frac{8}{15}m_2 \right] \mathbf{v}_1 \right\} \right\} \\ & + \frac{1}{c^7}\left\{ \frac{G^2 m_1 m_2}{r^3} \left\{ \left[ -46(\mathbf{n} \cdot \mathbf{v}_1)^5 + 214(\mathbf{n} \cdot \mathbf{v}_1)^4(\mathbf{n} \cdot \mathbf{v}_2) + 106(\mathbf{n} \cdot \mathbf{v}_1)^2(\mathbf{n} \cdot \mathbf{v}_2)^3 + \left( -42\mathbf{v}_1^2 \right. \right. \right. \\ & + \frac{284}{7}(\mathbf{v}_1 \cdot \mathbf{v}_2) \Big) (\mathbf{n} \cdot \mathbf{v}_2)^3 + \left( \frac{1238}{7}\mathbf{v}_1^2 - \frac{1032}{7}(\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{284}{7}\mathbf{v}_2^2 \right) (\mathbf{n} \cdot \mathbf{v}_1)(\mathbf{n} \cdot \mathbf{v}_2)^2 + \left( -282(\mathbf{n} \cdot \mathbf{v}_2)^2 \right. \\ & + 56\mathbf{v}_1^2 - \frac{536}{7}(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{46}{7}\mathbf{v}_2^2 \Big) (\mathbf{n} \cdot \mathbf{v}_1)^3 + \left( -\frac{1420}{7}\mathbf{v}_1^2 + \frac{1452}{7}(\mathbf{v}_1 \cdot \mathbf{v}_2) + 22\mathbf{v}_2^2 \right) (\mathbf{n} \cdot \mathbf{v}_1)^2(\mathbf{n} \cdot \mathbf{v}_2) \\ & + \left( \frac{86}{7}\mathbf{v}_1^4 - \frac{184}{5}\mathbf{v}_1^2(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{816}{35}(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + \frac{146}{35}\mathbf{v}_1^2\mathbf{v}_2^2 - \frac{104}{35}(\mathbf{v}_1 \cdot \mathbf{v}_2)\mathbf{v}_2^2 \right) (\mathbf{n} \cdot \mathbf{v}_2) + \left( 8(\mathbf{n} \cdot \mathbf{v}_2)^4 \right. \\ & \left. \left. - \frac{138}{35}\mathbf{v}_1^4 + \frac{932}{35}\mathbf{v}_1^2(\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{120}{7}(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - \frac{38}{5}\mathbf{v}_1^2\mathbf{v}_2^2 - \frac{188}{35}(\mathbf{v}_1 \cdot \mathbf{v}_2)\mathbf{v}_2^2 + \frac{52}{7}\mathbf{v}_2^4 \right) (\mathbf{n} \cdot \mathbf{v}_1) \right] \mathbf{n} \\ & + \left[ \frac{334}{7}(\mathbf{n} \cdot \mathbf{v}_1)^4 - \frac{752}{7}(\mathbf{n} \cdot \mathbf{v}_1)^3(\mathbf{n} \cdot \mathbf{v}_2) + \frac{584}{7}(\mathbf{n} \cdot \mathbf{v}_1)(\mathbf{n} \cdot \mathbf{v}_2)^3 - \frac{208}{7}(\mathbf{n} \cdot \mathbf{v}_2)^4 + \frac{1006}{105}\mathbf{v}_1^4 - \frac{2392}{105}\mathbf{v}_1^2(\mathbf{v}_1 \cdot \mathbf{v}_2) \right. \\ & + \frac{40}{7}(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 + \left( \frac{3748}{35}\mathbf{v}_1^2 - \frac{1296}{35}(\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{548}{7}\mathbf{v}_2^2 \right) (\mathbf{n} \cdot \mathbf{v}_1)(\mathbf{n} \cdot \mathbf{v}_2) - \frac{22}{105}\mathbf{v}_1^2\mathbf{v}_2^2 + \frac{1564}{105}(\mathbf{v}_1 \cdot \mathbf{v}_2)\mathbf{v}_2^2 - \frac{36}{5}\mathbf{v}_2^4 \\ & + \left( 6(\mathbf{n} \cdot \mathbf{v}_2)^2 - \frac{2356}{35}\mathbf{v}_1^2 + \frac{264}{5}(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{814}{35}\mathbf{v}_2^2 \right) (\mathbf{n} \cdot \mathbf{v}_1)^2 + \left( -\frac{1098}{35}\mathbf{v}_1^2 - \frac{60}{7}(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{276}{7}\mathbf{v}_2^2 \right) (\mathbf{n} \cdot \mathbf{v}_2)^2 \Big] \mathbf{v}_1 \\ & + \left[ -82(\mathbf{n} \cdot \mathbf{v}_1)^4 + \frac{1684}{7}(\mathbf{n} \cdot \mathbf{v}_1)^3(\mathbf{n} \cdot \mathbf{v}_2) + \frac{284}{7}(\mathbf{n} \cdot \mathbf{v}_1)(\mathbf{n} \cdot \mathbf{v}_2)^3 + \left( \frac{2542}{35}\mathbf{v}_1^2 - \frac{2512}{35}(\mathbf{v}_1 \cdot \mathbf{v}_2) \right) (\mathbf{n} \cdot \mathbf{v}_2)^2 \right. \\ & - \frac{614}{35}\mathbf{v}_1^4 + \frac{1868}{35}\mathbf{v}_1^2(\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{248}{7}(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - \frac{102}{7}\mathbf{v}_1^2\mathbf{v}_2^2 + \frac{496}{35}(\mathbf{v}_1 \cdot \mathbf{v}_2)\mathbf{v}_2^2 + \left( -\frac{6752}{35}\mathbf{v}_1^2 + \frac{1424}{7}(\mathbf{v}_1 \cdot \mathbf{v}_2) \right. \\ & \left. \left. - \frac{104}{35}\mathbf{v}_2^2 \right) (\mathbf{n} \cdot \mathbf{v}_1)(\mathbf{n} \cdot \mathbf{v}_2) + \left( -\frac{1394}{7}(\mathbf{n} \cdot \mathbf{v}_2)^2 + \frac{3912}{35}\mathbf{v}_1^2 - \frac{4852}{35}(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{646}{35}\mathbf{v}_2^2 \right) (\mathbf{n} \cdot \mathbf{v}_1)^2 \right] \mathbf{v}_2 \Big\} \\ & + \frac{G^3 m_1 m_2}{r^4} \left\{ \left[ m_1 \left( \frac{7428}{35}(\mathbf{n} \cdot \mathbf{v}_1)^3 - \frac{15492}{35}(\mathbf{n} \cdot \mathbf{v}_1)^2(\mathbf{n} \cdot \mathbf{v}_2) + \left( \frac{10156}{105}\mathbf{v}_1^2 - \frac{488}{5}(\mathbf{v}_1 \cdot \mathbf{v}_2) \right) (\mathbf{n} \cdot \mathbf{v}_2) \right. \right. \right. \\ & + \left( \frac{1128}{5}(\mathbf{n} \cdot \mathbf{v}_2)^2 - \frac{1828}{15}\mathbf{v}_1^2 + \frac{472}{3}(\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{3692}{105}\mathbf{v}_2^2 \right) (\mathbf{n} \cdot \mathbf{v}_1) \Big] + m_2 \left( \frac{10188}{35}(\mathbf{n} \cdot \mathbf{v}_1)^3 \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{23776}{35}(\mathbf{n} \cdot \mathbf{v}_1)^2(\mathbf{n} \cdot \mathbf{v}_2) - \frac{3272}{35}(\mathbf{n} \cdot \mathbf{v}_2)^3 + \left( \frac{16692}{35}(\mathbf{n} \cdot \mathbf{v}_2)^2 - \frac{18656}{105}\mathbf{v}_1^2 + \frac{9472}{35}(\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{10064}{105}\mathbf{v}_2^2 \right)(\mathbf{n} \cdot \mathbf{v}_1) \\
& + \left( \frac{5308}{35}\mathbf{v}_1^2 - \frac{22576}{105}(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{6896}{105}\mathbf{v}_2^2 \right)(\mathbf{n} \cdot \mathbf{v}_2) \Big] \mathbf{n} + \left[ m_1 \left( -\frac{3516}{35}(\mathbf{n} \cdot \mathbf{v}_1)^2 + \frac{2528}{21}(\mathbf{n} \cdot \mathbf{v}_1)(\mathbf{n} \cdot \mathbf{v}_2) \right. \right. \\
& - \frac{152}{15}(\mathbf{n} \cdot \mathbf{v}_2)^2 + \frac{84}{5}\mathbf{v}_1^2 - \frac{2672}{105}(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{932}{105}\mathbf{v}_2^2 \Big) + m_2 \left( -\frac{17308}{105}(\mathbf{n} \cdot \mathbf{v}_1)^2 + \frac{1204}{5}(\mathbf{n} \cdot \mathbf{v}_1)(\mathbf{n} \cdot \mathbf{v}_2) \right. \\
& - \frac{1184}{15}(\mathbf{n} \cdot \mathbf{v}_2)^2 + \frac{4394}{105}\mathbf{v}_1^2 - \frac{7852}{105}(\mathbf{v}_1 \cdot \mathbf{v}_2) + \frac{1194}{35}\mathbf{v}_2^2 \Big) \Big] \mathbf{v}_1 + \left[ m_1 \left( \frac{9356}{105}(\mathbf{n} \cdot \mathbf{v}_1)^2 \right. \right. \\
& - \frac{488}{5}(\mathbf{n} \cdot \mathbf{v}_1)(\mathbf{n} \cdot \mathbf{v}_2) - \frac{284}{35}\mathbf{v}_1^2 + \frac{128}{15}(\mathbf{v}_1 \cdot \mathbf{v}_2) \Big) + m_2 \left( \frac{16052}{105}(\mathbf{n} \cdot \mathbf{v}_1)^2 - \frac{1524}{7}(\mathbf{n} \cdot \mathbf{v}_1)(\mathbf{n} \cdot \mathbf{v}_2) \right. \\
& + \frac{7256}{105}(\mathbf{n} \cdot \mathbf{v}_2)^2 - \frac{3358}{105}\mathbf{v}_1^2 + \frac{6028}{105}(\mathbf{v}_1 \cdot \mathbf{v}_2) - \frac{2726}{105}\mathbf{v}_2^2 \Big) \Big] \mathbf{v}_2 \Big\} + \frac{G^4 m_1 m_2}{r^5} \left\{ \left[ \frac{8}{35}m_1^2(\mathbf{n} \cdot \mathbf{v}_1) \right. \right. \\
& + m_2^2 \left( -\frac{152}{15}(\mathbf{n} \cdot \mathbf{v}_1) + \frac{164}{15}(\mathbf{n} \cdot \mathbf{v}_2) \right) + m_1 m_2 \left( -\frac{96}{7}(\mathbf{n} \cdot \mathbf{v}_1) + 12(\mathbf{n} \cdot \mathbf{v}_2) \right) \Big] \mathbf{n} \\
& + \left[ -\frac{8}{105}m_1^2 - \frac{394}{105}m_1 m_2 - \frac{386}{105}m_2^2 \right] \mathbf{v}_1 + \left[ \frac{162}{35}m_1 m_2 + \frac{382}{105}m_2^2 \right] \mathbf{v}_2 \Big\} \Big\}, \tag{5.6a}
\end{aligned}$$

$$\ddot{\mathbf{x}}_2 = (1 \rightleftharpoons 2). \tag{5.6b}$$

In the next subsection we show the second way to compute the equations above.

### C. Second method to calculate the Euler-Lagrangian equations of motions

The second route is less straightforward, but leads to the confirmation of the preceding result. We start from the Hamiltonian equations (2.3) for  $\dot{\mathbf{p}}_a$ :

$$\dot{\mathbf{p}}_a = -\frac{\partial H}{\partial \mathbf{x}_a}, \tag{5.7}$$

where we replace  $\dot{\mathbf{p}}_a$  by the derivative of the right-hand side of Eq. (5.3). The formal time derivatives of  $\dot{\mathbf{x}}_1$  and  $\dot{\mathbf{x}}_2$ , denoted by  $\mathbf{a}_1$  and  $\mathbf{a}_2$  respectively, are kept unevaluated. The right-hand side of Eq. (5.7) involve the reaction contributions Eqs. (4.3), as well as the Newtonian and post-Newtonian parts (3.7). We systematically eliminate the particle momenta in favor of the particle coordinate velocities by means of Eqs. (5.3) (without  $1/c^7$ ). To calculate  $\ddot{\mathbf{x}}_1$ , we consider the coefficient of  $\mathbf{a}_1$  in  $\dot{\mathbf{p}}_1$ , and invert it up to the order  $1/c^7$ . The equation of motion (5.7) in its present form is multiplied by the latter result. Next, we gather all terms but  $\mathbf{a}_1$  in the right-hand side. Simultaneously, the acceleration  $\ddot{\mathbf{x}}_2$  is isolated in the same way. We arrive at the coupled system:

$$\mathbf{a}_1 = f(\mathbf{x}_1, \mathbf{x}_2, \dot{\mathbf{x}}_1 = \mathbf{v}_1, \dot{\mathbf{x}}_2 = \mathbf{v}_2, \ddot{\mathbf{x}}_1 = \mathbf{a}_1, \ddot{\mathbf{x}}_2 = \mathbf{a}_2), \tag{5.8a}$$

$$\mathbf{a}_2 = f(\mathbf{x}_1, \mathbf{x}_2, \dot{\mathbf{x}}_1 = \mathbf{v}_1, \dot{\mathbf{x}}_2 = \mathbf{v}_2, \ddot{\mathbf{x}}_1 = \mathbf{a}_1, \ddot{\mathbf{x}}_2 = \mathbf{a}_2). \tag{5.8b}$$

The various contributions are sorted in powers of  $1/c$ , and labeled with two different indices. The second index refers to the  $1/c$  exponent, whereas the first index identifies the body;  $N$  stands for the Newtonian part.

$$\mathbf{a}_1 = N_{10} + \frac{1}{c^2}f_{12}(\mathbf{a}_1, \mathbf{a}_2) + \frac{1}{c^5}f_{15}(\mathbf{a}_1, \mathbf{a}_2) + \frac{1}{c^7}f_{17}(\mathbf{a}_1, \mathbf{a}_2), \tag{5.9a}$$

$$\mathbf{a}_2 = N_{20} + \frac{1}{c^2}f_{22}(\mathbf{a}_1, \mathbf{a}_2) + \frac{1}{c^5}f_{25}(\mathbf{a}_1, \mathbf{a}_2) + \frac{1}{c^7}f_{27}(\mathbf{a}_1, \mathbf{a}_2). \tag{5.9b}$$

The decoupling is achieved by iterative substitution of Eq. (5.9a) into Eq. (5.9b) and vice versa. We get

$$\mathbf{a}_1 = N_{10} + \frac{1}{c^2}g_{12}(\mathbf{a}_1) + \frac{1}{c^5}g_{15}(\mathbf{a}_1) + \frac{1}{c^7}g_{17}(\mathbf{a}_1), \tag{5.10a}$$

$$\mathbf{a}_2 = N_{20} + \frac{1}{c^2}g_{22}(\mathbf{a}_2) + \frac{1}{c^5}g_{25}(\mathbf{a}_2) + \frac{1}{c^7}g_{27}(\mathbf{a}_2). \tag{5.10b}$$

Next, we compute  $\mathbf{a}_1$  and  $\mathbf{a}_2$  iteratively, and insert the result into the right-hand sides of Eqs. (5.10). The acceleration has finally the following structure

$$\mathbf{a}_1 = f(\mathbf{x}_1, \mathbf{x}_2, \dot{\mathbf{x}}_1 = \mathbf{v}_1, \dot{\mathbf{x}}_2 = \mathbf{v}_2), \tag{5.11a}$$

$$\mathbf{a}_2 = f(\mathbf{x}_1, \mathbf{x}_2, \dot{\mathbf{x}}_1 = \mathbf{v}_1, \dot{\mathbf{x}}_2 = \mathbf{v}_2). \tag{5.11b}$$

We recover Eqs. (5.6).

In order to confirm the correctness of expressions above for  $\ddot{\mathbf{x}}_1$  and  $\ddot{\mathbf{x}}_2$ , we shall check it yields the correct energy loss (3.18). For this goal, we begin by deriving the reaction forces with the help of Eqs. (5.6)

$$\mathbf{F}_1^{\text{reac}} = m_1 \ddot{\mathbf{x}}_1, \tag{5.12a}$$

$$\mathbf{F}_2^{\text{reac}} = m_2 \ddot{\mathbf{x}}_2. \quad (5.12b)$$

After multiplying these relations by  $\dot{\mathbf{x}}_1 = \mathbf{v}_1$  or  $\dot{\mathbf{x}}_2 = \mathbf{v}_2$  respectively, we sum to obtain

$$m_1 (\ddot{\mathbf{x}}_1 \cdot \mathbf{v}_1) + m_2 (\ddot{\mathbf{x}}_2 \cdot \mathbf{v}_2) = h(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2) \quad (5.13)$$

This is not yet the instantaneous energy loss due particularly to the presence of the Newtonian and post-Newtonian terms coming from Eqs. (5.6). We then must search for the function  $g = g(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2)$  to be added to  $h$  as defined in Eq. (5.13) in order to isolate the radiative 2.5PN and 3.5PN parts, which does not only amount to removing the Newtonian and post-Newtonian contributions. We shall then be able to find the average power by substituting to  $\mathbf{v}^2$  its value (3.16).

The correct energy loss is given by the time derivative of  $\mathcal{E} = H(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1(\mathbf{x}_a, \mathbf{v}_a), \mathbf{p}_2(\mathbf{x}_a, \mathbf{v}_a))$  truncated at the 1PN level. The particle momenta of the 1PN Hamiltonian (4.4) are expressed by means of the particle coordinate velocities (5.3) up to the order  $1/c^5$  (only). When performing the differentiation of the resulting energy with respect to time,  $\dot{\mathbf{x}}_1 = \mathbf{a}_1$  and  $\dot{\mathbf{x}}_2 = \mathbf{a}_2$  — coming from the derivatives of  $\dot{\mathbf{x}}_1 = \mathbf{v}_1$  and  $\dot{\mathbf{x}}_2 = \mathbf{v}_2$  — are replaced by expressions (5.6) except in the Newtonian term  $m_1 (\mathbf{a}_1 \cdot \mathbf{v}_1) + m_2 (\mathbf{a}_2 \cdot \mathbf{v}_2)$  where they are left unevaluated. The function  $g = g(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2)$  is just the difference  $d\mathcal{E}/dt - m_1 (\mathbf{a}_1 \cdot \mathbf{v}_1) - m_2 (\mathbf{a}_2 \cdot \mathbf{v}_2)$ .

By construction,  $h + g$  contains no Newtonian or post-Newtonian terms. It actually represents the instantaneous luminosity  $\mathcal{L}_{\leq 3.5\text{PN}}^{(3)} = -d\mathcal{E}/dt$ , we intended to check, given by the negative instantaneous energy loss.<sup>5</sup> The result of the averaging procedure in the case of quasi-elliptic orbital motion is in agreement with our preceding results and with Eqs. (4.20) and (4.21) of Ref. [26].

## VI. COMPARISM WITH IYER-WILL FORMALISM

In this section, we shall show that the dissipative parts of Eqs. (5.6) are compatible with the generic reaction

<sup>5</sup> The full result for  $\langle \mathcal{L}_{\leq 3.5\text{PN}}^{(3)} \rangle$  is presented in Ref. [24].

force of Iyer and Will [2] depending on  $(2 + 6)$  gauge parameters, as the ones given in the Eqs. (4.1a)–(4.1d) in Ref. [2]. The first two parameters at the 2.5PN order and the six coefficients at the 3.5PN order correspond to the gauge freedom.

As a matter of fact, it is sufficient to prove that for certain values of the unknowns, the latter reaction force coincides with ours. This requires to compute the relative acceleration

$$\mathbf{a} = \mathbf{a}_{12} := \mathbf{a}_1 - \mathbf{a}_2 = \ddot{\mathbf{x}}_1 - \ddot{\mathbf{x}}_2 \quad (6.1)$$

from Eq. (5.5) for  $\ddot{\mathbf{x}}_1 = f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2)$  and  $\ddot{\mathbf{x}}_2 = f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2)$  respectively and to go to the center-of-mass frame

$$\mathbf{p}_1 + \mathbf{p}_2 = 0. \quad (6.2)$$

As the accelerations we start with are function of  $\mathbf{p}_1$  and  $\mathbf{p}_2$  rather than  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , we must simply set  $\mathbf{p}_1$  to  $\mathbf{p}$  and  $\mathbf{p}_2$  to  $-\mathbf{p}$  in  $\mathbf{a} = \mathbf{a}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2)$  [see Eq. (6.1)].

The next step consists in establishing the link between the particle momenta  $\mathbf{p}$  and the relative velocity  $\mathbf{v} = \mathbf{v}_{12} := \mathbf{v}_1 - \mathbf{v}_2$  in order to eliminate the particle momenta from  $\mathbf{a} = \mathbf{a}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p})$ . We thus restore the Newtonian and post-Newtonian part in Eqs. (4.2) and calculate the relative velocity

$$\mathbf{v} = \mathbf{v}_{12} := \mathbf{v}_1 - \mathbf{v}_2 = \dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2. \quad (6.3)$$

We make the substitutions  $\mathbf{p}_1 \rightarrow \mathbf{p}$  and  $\mathbf{p}_2 \rightarrow -\mathbf{p}$  in  $\mathbf{v} = \mathbf{v}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2)$ . The relation  $\mathbf{v} = \mathbf{v}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p})$  can be inverted with the aid of the iterative method described above to get the momentum as a function of the relative velocity  $\mathbf{p} = \mathbf{p}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v})$ . Inserting  $\mathbf{p} = \mathbf{p}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v})$  in  $\mathbf{a} = \mathbf{a}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p})$  leads to  $\mathbf{a} = \mathbf{a}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v})$ . After introducing the total mass  $M$  as well as the dimensionless mass parameter  $\nu$  through Eqs. (3.10) and  $\mu = \nu M$ , the relative acceleration reads [24]:

$$\begin{aligned} \mathbf{a} = & -\frac{GM}{r^2} \mathbf{n} + \frac{1}{c^2} \left\{ \frac{GM}{r^2} \left\{ \left[ -(1+3\nu) \mathbf{v}^2 + \frac{3}{2} \dot{r}^2 \nu \right] \mathbf{n} + (4\dot{r} - 2\dot{r}\nu) \mathbf{v} \right\} + \frac{G^2 M^2}{r^3} (4 + 2\nu) \mathbf{n} \right\} \\ & + \frac{1}{c^5} \left\{ \frac{G^2 M^2}{r^3} \left[ \left( -24\dot{r}^3 \nu + \frac{96}{5} \dot{r} \mathbf{v}^2 \nu \right) \mathbf{n} + \left( \frac{64}{5} \dot{r}^2 \nu - \frac{88}{15} \mathbf{v}^2 \nu \right) \mathbf{v} \right] + \frac{G^3 M^3}{r^4} \left( \frac{16}{5} \dot{r} \nu \mathbf{n} - \frac{8}{15} \nu \mathbf{v} \right) \right\} \\ & + \frac{1}{c^7} \left\{ \frac{G^2 M^2}{r^3} \left\{ \left[ -46\dot{r}^5 \nu + 24\dot{r}^5 \nu^2 + \mathbf{v}^4 \left( -\frac{138}{35} \dot{r} \nu - \frac{516}{35} \dot{r} \nu^2 \right) + \mathbf{v}^2 \left( 56\dot{r}^3 \nu - \frac{4}{7} \dot{r}^3 \nu^2 \right) \right] \mathbf{n} \right. \right. \\ & \left. \left. + \left[ \frac{334}{7} \dot{r}^4 \nu - \frac{268}{7} \dot{r}^4 \nu^2 + \mathbf{v}^4 \left( \frac{1006}{105} \nu - \frac{64}{105} \nu^2 \right) + \mathbf{v}^2 \left( -\frac{2356}{35} \dot{r}^2 \nu + \frac{148}{5} \dot{r}^2 \nu^2 \right) \right] \mathbf{v} \right\} \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{G^3 M^3}{r^4} \left\{ \left[ \frac{10188}{35} \dot{r}^3 \nu + \frac{324}{7} \dot{r}^3 \nu^2 + \mathbf{v}^2 \left( -\frac{18656}{105} \dot{r} \nu - \frac{1116}{35} \dot{r} \nu^2 \right) \right] \mathbf{n} + \left[ -\frac{17308}{105} \dot{r}^2 \nu - \frac{244}{21} \dot{r}^2 \nu^2 \right. \right. \\
& \left. \left. + \mathbf{v}^2 \left( \frac{4394}{105} \nu - \frac{16}{35} \nu^2 \right) \right] \mathbf{v} \right\} + \frac{G^4 M^4}{r^5} \left[ \left( -\frac{152}{15} \dot{r} \nu - \frac{632}{105} \dot{r} \nu^2 \right) \mathbf{n} - \left( \frac{386}{105} \nu + \frac{16}{15} \nu^2 \right) \mathbf{v} \right] \right\} \quad (6.4)
\end{aligned}$$

with the notation

$$\dot{r} = (\mathbf{n} \cdot \mathbf{v}) = (\mathbf{n}_{12} \cdot \mathbf{v}_{12}) . \quad (6.5)$$

This result corresponds to the generic equation of motion (1.4) and Eqs. (2) in Ref. [2] by Iyer and Will (in the following equations we take  $G = 1 = c$ ):

$$\begin{aligned}
\frac{d^2 \mathbf{r}}{dt^2} = & -\frac{M}{r^2} \mathbf{n} + \frac{M}{r^2} \left[ \mathbf{n} (A_{1\text{PN}} + A_{2\text{PN}} + A_{3\text{PN}}) + \dot{r} \mathbf{v} (B_{1\text{PN}} + B_{2\text{PN}} + B_{3\text{PN}}) \right] \\
& + \frac{8}{5} \nu \frac{M}{r^2} \frac{M}{r} \left[ \dot{r} \mathbf{n} (A_{2.5\text{PN}} + A_{3.5\text{PN}}) - \mathbf{v} (B_{2.5\text{PN}} + B_{3.5\text{PN}}) \right] \quad (6.6)
\end{aligned}$$

with [see Eqs. (2.2a), (2.2b), (2.5) and (2.12) in Ref. [2]]

$$A_{1\text{PN}} = -(1 + 3\nu) \mathbf{v}^2 + \frac{3}{2} \nu \dot{r}^2 + 2(2 + \nu) \frac{M}{r}, \quad (6.7a)$$

$$B_{1\text{PN}} = 2(2 - \nu), \quad (6.7b)$$

$$A_{2.5\text{PN}} = a_1 \mathbf{v}^2 + a_2 \frac{M}{r} + a_3 \dot{r}^2, \quad (6.7c)$$

$$B_{2.5\text{PN}} = b_1 \mathbf{v}^2 + b_2 \frac{M}{r} + b_3 \dot{r}^2, \quad (6.7d)$$

$$\begin{aligned}
A_{3.5\text{PN}} = & c_1 \mathbf{v}^4 + c_2 \mathbf{v}^2 \frac{M}{r} + c_3 \mathbf{v}^2 \dot{r}^2 + c_4 \dot{r}^2 \frac{M}{r} \\
& + c_5 \dot{r}^4 + c_6 \frac{M^2}{r^2}, \quad (6.7e)
\end{aligned}$$

$$\begin{aligned}
B_{3.5\text{PN}} = & d_1 \mathbf{v}^4 + d_2 \mathbf{v}^2 \frac{M}{r} + d_3 \mathbf{v}^2 \dot{r}^2 + d_4 \dot{r}^2 \frac{M}{r} \\
& + d_5 \dot{r}^4 + d_6 \frac{M^2}{r^2}. \quad (6.7f)
\end{aligned}$$

Note that we are not interested in the 2PN and 3PN terms in the present work.

Iyer and Will establish in Refs. [2, 6] that the energy and angular momentum balances hold if and only if the coefficients  $a_i, b_i, c_i$  and  $d_i$  satisfies the following relations:

$$a_1 = 3 + 3\beta, \quad a_2 = \frac{23}{3} + 2\alpha - 3\beta, \quad a_3 = -5\beta, \quad (6.8a)$$

$$b_1 = 2 + \alpha, \quad b_2 = 2 - \alpha, \quad b_3 = -3 - 3\alpha \quad (6.8b)$$

for the 2.5PN coefficients [see Eqs. (2.11) of Ref. [2]], and

$$c_1 = \frac{1}{28} (117 + 132\nu) - \frac{3}{2} \beta (1 - 3\nu) + 3\delta_2 - 3\delta_6, \quad (6.9a)$$

$$c_2 = -\frac{1}{42} (297 - 310\nu) - 3\alpha (1 - 4\nu) - \frac{3}{2} \beta (7 + 13\nu)$$

$$-2\delta_1 - 3\delta_2 + 3\delta_5 + 3\delta_6, \quad (6.9b)$$

$$c_3 = \frac{5}{28} (19 - 72\nu) + \frac{5}{2} \beta (1 - 3\nu) - 5\delta_2 + 5\delta_4 + 5\delta_6, \quad (6.9c)$$

$$c_4 = -\frac{1}{28} (687 - 368\nu) - 6\alpha\nu + \frac{1}{2} \beta (54 + 17\nu) - 2\delta_2 - 5\delta_4 - 6\delta_5, \quad (6.9d)$$

$$c_5 = -7\delta_4, \quad (6.9e)$$

$$c_6 = -\frac{1}{21} (1533 + 498\nu) - \alpha (14 + 9\nu) + 3\beta (7 + 4\nu) - 2\delta_3 - 3\delta_5, \quad (6.9f)$$

$$d_1 = -3(1 - 3\nu) - \frac{3}{2} \alpha (1 - 3\nu) - \delta_1, \quad (6.9g)$$

$$d_2 = -\frac{1}{84} (139 + 768\nu) - \frac{1}{2} \alpha (5 + 17\nu) + \delta_1 - \delta_3, \quad (6.9h)$$

$$d_3 = \frac{1}{28} (369 - 624\nu) + \frac{3}{2} (3\alpha + 2\beta) (1 - 3\nu) + 3\delta_1 - 3\delta_6, \quad (6.9i)$$

$$d_4 = \frac{1}{42} (295 - 335\nu) + \frac{1}{2} \alpha (38 - 11\nu) - 3\beta (1 - 3\nu) + 2\delta_1 + 4\delta_3 + 3\delta_6, \quad (6.9j)$$

$$d_5 = \frac{5}{28} (19 - 72\nu) - 5\beta (1 - 3\nu) + 5\delta_6, \quad (6.9k)$$

$$d_6 = -\frac{1}{21} (634 - 66\nu) + \alpha (7 + 3\nu) + \delta_3 \quad (6.9l)$$

for the 3.5PN coefficients [see Eqs. (2.18) of Ref. [2]].<sup>6</sup>

The two parameters  $\alpha, \beta$  at the 2.5PN order and the six coefficients  $\delta_i$  at the 3.5PN order correspond to the gauge freedom and have no physical meaning. As mentioned in Ref. [8], at the 2.5PN order, the values

<sup>6</sup> In Ref. [2] the notation  $\varepsilon_5$  was used in place of  $\delta_6$ .

$\alpha = -1$ ,  $\beta = 0$  characterize the gauge of Damour and Deruelle [28], while  $\alpha = 4$ ,  $\beta = 5$  correspond to the so-called Burke-Thorne gauge [see for example §36.11 of [29]] also used by Blanchet [7].

It is then a non-trivial check to verify that the 18 coefficients  $a_i$ ,  $b_i$ ,  $c_i$  and  $d_i$  parameterizing our 2.5PN and 3.5PN terms [Eqs. (6.7c)–(6.7f) specialized for Eq. (6.4)] yield a unique, self-consistent solution for the 8 gauge coefficients. They are [24]:

for the 2.5PN order:

$$\alpha = \frac{5}{3}, \quad \beta = 3; \quad (6.10a)$$

for the 3.5PN order:

$$\delta_1 = \frac{41}{84} + \frac{677}{42}\nu, \quad \delta_2 = -\frac{61}{14} - \frac{151}{14}\nu, \quad (6.10b)$$

$$\delta_3 = \frac{583}{28} - \frac{157}{21}\nu, \quad \delta_4 = \frac{115}{28} - \frac{15}{7}\nu, \quad (6.10c)$$

$$\delta_5 = -\frac{961}{42} + \frac{16}{3}\nu, \quad \delta_6 = -\frac{51}{14} - \frac{23}{14}\nu. \quad (6.10d)$$

## VII. DISCUSSION

In this paper, we have computed PN reactive contributions to the ADM Hamiltonian for a pair of compact objects moving in general orbits, using existing results available for the post-Newtonian  $N$ -body Hamiltonian, computed by Jaranowski and Schäfer [1]. The two-body Hamiltonian is employed to compute the first PN corrections to the instantaneous gravitational energy loss associated with inspiraling compact binaries. We also derive PN corrections to reactive equations of motion via Hamiltonian and Euler-Lagrangian methods. The expressions for PN reactive accelerations, computed in the ADM gauge, are consistent with those of Iyer and Will. Employing the various reactive equations of motion we also computed instantaneous gravitational wave luminosity for the compact binary in general orbits. These luminosities are in total agreement with their orbital averaged counterparts in the case of quasi-elliptical orbital motion.

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## APPENDIX A: HADAMARD REGULARIZATION

Let  $f$  be a smooth real-valued function defined on  $\mathbb{R}^3$  deprived of a point  $\mathbf{x}_0 \in \mathbb{R}^3$ , where it may be singular. We consider the family of auxiliary functions  $f_{\mathbf{n}}(\varepsilon) := f(\mathbf{x}_0 + \varepsilon\mathbf{n})$ , labeled by unit vectors  $\mathbf{n}$ . We expand  $f_{\mathbf{n}}$  into Laurent series around  $\varepsilon = 0$ :

$$f_{\mathbf{n}}(\varepsilon) = \sum_{m=-N}^{\infty} a_m(\mathbf{n}) \varepsilon^m. \quad (A1)$$

The coefficients  $a_m$  of this expansion depend on the unit vector  $\mathbf{n}$ . We define the regularized value of the function  $f$  at  $\mathbf{x}_0$  as the coefficient of  $\varepsilon^0$  in the expansion (A1) averaged over all directions:

$$f_{\text{reg}}(\mathbf{x}_0) := \frac{1}{4\pi} \oint a_0(\mathbf{n}) d\Omega, \quad (A2)$$

where  $d\Omega$  is the elementary solid angle viewed from the point  $\mathbf{x}_0$ . This procedure is called Hadamard regularization. To calculate the integrals in Eq. (A2) we used the following equations

$$\frac{1}{4\pi} \oint n^{i_1} n^{i_2} \dots n^{i_l} d\Omega = 0, \quad \text{for odd } l, \quad (A3a)$$

$$\frac{1}{4\pi} \oint n^{i_1} n^{i_2} \dots n^{i_{2p}} d\Omega = \frac{1}{(2p+1)!!} \underbrace{\delta^{\{i_1 i_2 \dots i_{2p-1} i_{2p}\}}}_{(2p-1)!! \text{ terms}}, \quad (A3b)$$

with  $(2p+1)!! = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2p+1)$ . The curly braces in Eq. (A3b) mean the symmetric combination of the indices:

$$\delta^{\{ij} \delta^{kl\}} = \underbrace{\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}}_{3!! \text{ terms}}. \quad (A4)$$

During the identification process of the primed and unprimed variables,  $\mathbf{x}_{1'} \rightarrow \mathbf{x}_1$  and  $\mathbf{x}_{2'} \rightarrow \mathbf{x}_2$ , in the calculation of the derivation of the Hamiltonians given by the Eqs. (2.6) and (2.8), some singularities appear. The purpose of the regularization is the elimination of these singularities. It is a matter of choice whether we apply first the regularization for index  $1'$  and then for index  $2'$ , or the other way around.

As an example, we shall apply Hadamard regularization procedure on the term

$$\frac{(\mathbf{n}_{11'} \cdot \mathbf{n}_{22'}) (\mathbf{n}_{11'} \cdot \mathbf{p}_1) (\mathbf{n}_{22'} \cdot \mathbf{p}_2)}{r_{11'}^2 r_{1'2'}^3} = n_{11'}^i n_{22'}^j n_{11'}^j p_1^i n_{22'}^k p_2^k r_{11'}^{-2} r_{1'2'}^{-3}. \quad (A5)$$

We intend to identify index 1 with  $1'$  as a first step. We start by calculating the limit for  $\varrho_1^i = -r_{11'}^i \rightarrow 0$  after the replacement:

$$r_{1'2'}^i = r_{12'}^i - r_{11'}^i = r_{12'}^i + \varrho_1^i. \quad (A6)$$

in Eq. (A5). To this purpose, we expand

$$\begin{aligned} r_{1'2'}^m &= f(r_{12'}^i + \varrho_2^i) \\ &= \sum_{l=0}^{+\infty} \frac{\varrho_2^l}{l!} \partial_L r_{12'}^m = \sum_{l=0}^{+\infty} \frac{(-1)^l}{l!} r_{11'}^l n_{11'}^L \partial_L r_{12'}^m. \end{aligned} \quad (\text{A7})$$

We now perform the first angular integration over the terms of Eq. (A5) depending on the variable labeled by  $1'$ . Here is  $m = -3$  and  $l = 2$ .

$$\begin{aligned} &\frac{1}{4\pi} \oint n_{11'}^i n_{11'}^j r_{11'}^{-2} r_{1'2'}^{-3} d\Omega_{11'} \\ &= \sum_{l=0}^{+\infty} \frac{(-1)^l}{l!} r_{11'}^{l-2} \partial_{(12')L} r_{12'}^{-3} \frac{1}{4\pi} \oint n_{11'}^i n_{11'}^j n_{11'}^L d\Omega_{11'} \\ &= \frac{1}{2} \partial_{(12')ab} r_{12'}^{-3} \frac{1}{4\pi} \oint n_{11'}^i n_{11'}^j n_{11'}^a n_{11'}^b d\Omega_{11'} \\ &= \frac{1}{2} \partial_{(12')ab} r_{12'}^{-3} \frac{1}{15} (\delta^{ij} \delta^{ab} + \delta^{ia} \delta^{jb} + \delta^{ib} \delta^{ja}) \\ &= \frac{1}{30} (2\partial_{(12')ij} r_{12'}^{-3} + \delta^{ij} \Delta_{(12')} r_{12'}^{-3}). \end{aligned} \quad (\text{A8})$$

By using

$$\partial_{ij} r^m = [m(m-2)n^i n^j + m\delta^{ij}] r^{m-2}, \quad (\text{A9a})$$

$$\Delta r^m = m(m+1) r^{m-2}, \quad (\text{A9b})$$

we get

$$\frac{1}{4\pi} \oint n_{11'}^i n_{11'}^j r_{11'}^{-2} r_{1'2'}^{-3} d\Omega_{11'} = \frac{n_{12'}^i n_{12'}^j}{r_{12'}^5}. \quad (\text{A10})$$

In further computations we replace the terms on the right-hand side of Eq. (A10) with

$$r_{12'}^i = r_{12}^i + r_{22'}^i = r_{12}^i + \varrho_2^i, \quad (\text{A11a})$$

$$n_{12'}^i = \frac{r_{12}}{r_{12'}} n_{12}^i + \frac{r_{22'}}{r_{12'}} n_{22'}^i = \frac{r_{12}}{r_{12'}} n_{12}^i + \frac{\varrho_2}{r_{12'}} n_{22'}^i. \quad (\text{A11b})$$

The second step will consist in identifying index 2 with  $2'$ . We calculate the limit for  $\varrho_2^i = r_{22'}^i \rightarrow 0$ . The second angular integration over the finite part terms (*i.e.* the terms that are of order zero in power of  $\varrho_2^i$ ) depending on the variable labeled by  $2'$  can be performed immediately because there are no singularities in Eqs. (A11) for  $\varrho_2^i \rightarrow 0$ :

$$\begin{aligned} &\frac{n_{12}^i n_{12}^j p_1^j p_2^k}{r_{12}^5} \left( \frac{1}{4\pi} \oint n_{22'}^i n_{22'}^k d\Omega_{22'} \right) \\ &= \frac{n_{12}^i n_{12}^j p_1^j p_2^k}{r_{12}^5} \left( \frac{1}{3} \delta^{ik} \right). \end{aligned} \quad (\text{A12})$$

The result is

$$\begin{aligned} &\left[ \frac{(\mathbf{n}_{11'} \cdot \mathbf{n}_{22'}) (\mathbf{n}_{11'} \cdot \mathbf{p}_1) (\mathbf{n}_{22'} \cdot \mathbf{p}_2)}{r_{11'}^2 r_{1'2'}^3} \right]_{\left[ \begin{smallmatrix} \mathbf{x}_{1'} \rightarrow \mathbf{x}_1 \\ \mathbf{x}_{2'} \rightarrow \mathbf{x}_2 \end{smallmatrix} \right]} \\ &= \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1) (\mathbf{n}_{12} \cdot \mathbf{p}_2)}{3r_{12}^5}. \end{aligned} \quad (\text{A13})$$

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